Topology Qualifying Exam May 25, 2010
Mathematics Program CUNY Graduate Center

Instructions: Do at least two problems from each part, and at least eight problems overall.

Part I

1. Let $\{X_\lambda\}$ be a family of spaces, $p_\alpha : \prod X_\lambda \to X_\alpha$ any one of the projection maps.
   (a) Is p_α an open map? Explain.
   (b) Is p_α a closed map? Explain.
   (c) Is p_α a quotient map? Explain.

2. Let $f : X \to Y$ be a continuous map that is onto.
   (a) Does X connected imply Y connected? Explain.
   (b) Does X contractible imply Y contractible? Explain.
   (c) Does X metrizable imply Y metrizable? Explain.

3. (a) Prove: X compact and Hausdorff implies X is normal.
   (b) Give an example of a compact space which is *not* normal. Explain.
   (c) Give an example of a Hausdorff space which is *not* normal. Explain.

4. Prove either the Baire category theorem or the Tychynoff theorem.

Part II

1. (a) Compute $\pi_1(K)$, where K is the Klein bottle, and prove that $\pi_1(K)$ has a subgroup isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ by showing that there is a covering $T^2 \to K$, where T^2 is the 2-torus (orientable surface of genus 1).

   (b) Descibe the universal covering space of K and prove that any map $S^2 \to K$ is nullhomotopic.

2. Let X and Y be spaces with basepoints $x_0 \in X$, $y_0 \in Y$ and suppose $X \times Y$ is simply connected.

   (a) Prove: X and Y are both simply connected.

   (b) Prove: $X \vee Y$ is simply connected where $X \vee Y$ is the one point union of X and Y. Assume there are open neighborhoods $U \subset X$, $V \subset Y$ of x_0, y_0, respectively such that U is contractible, rel x_0, to x_0 and V is contractible, rel y_0, to y_0.

3. (a) Find a space whose fundamental group is free on n generators.

   (b) Using covering space theory, prove that every subgroup of the free group on n generators is a free group.

4. Let $S^1 \subset \mathbf{R}^3$ be an embedding (inclusion of a subspace) which is 'unknotted', for example take S^1 to be the unit vectors in the xy-plane. Compute $\pi_1(\mathbf{R}^3 - S^1)$.

Part III

1. Let X and Y be CW-complexes with basepoints $x_0 \in X$, $y_0 \in Y$. Suppose that the reduced homology of $X \times Y$ is trivial. Prove that the reduced homology of X, Y, and $X \vee Y$ are trivial.

2. Let $X = \mathbf{C}P^3 \times S^2$ and $Y = (S^2 \vee S^4 \vee S^6) \times S^2$. Are X and Y homotopy equivalent? Why or why not?

3. Prove that an even dimensional sphere cannot be a topological group (a *topological group* is a group G which is a topological space such that the map $G \times G \to G$ given by $(g, h) \mapsto gh^{-1}$ is continuous).

4. Compute $H_i(\mathbf{R}P^3 \times T^2; A)$ and $H^i(\mathbf{R}P^3 \times T^2; A)$ as abelian groups, for all $i \geq 0$, when $A = \mathbf{Z}$, $\mathbf{Q}$, $\mathbf{Z}/2\mathbf{Z}$, and $\mathbf{Z}/3\mathbf{Z}$. Here T^2 is the 2-torus.