Topology Qualifying Exam May 19, 2009
Mathematics Program CUNY Graduate Center

Instructions: Do at least two problems from each part, and at least eight problems overall.

Part I

1. Let X be a topological space.

   Define a relation on the points of X by $x \sim y$ if there exists a connected subset $K \subset X$ such that $x, y \in K$.

   (a) Prove that the relation is an equivalence relation.

   Equivalence classes are called **components** of X.

   (b) Prove that every component is connected.

   (c) Prove that every component is closed.

   (d) Prove or disprove that every component is open.

2. (a) Define what a quotient map is, and the quotient topology.

   (b) Let X be a topological space, $f : X \to Y$ a quotient map, and A an open subspace of X. Set $B = f(A)$, with the subspace topology of Y.

   Prove that if f is an open map, then the restriction of f is a quotient map $A \to B$.

   (A continuous function is **open** if the image of every open set is open.)

   (c) Give an example of a quotient map $f : X \to Y$ which is not open.

3. (a) State the Tietze Extension Theorem.

   (b) Let X be a normal space, $A \subset X$ a closed subspace, and $f : A \to S^1$ a continuous map (S^1 is the unit circle).

   Prove that there exist an open set $U \subset X$ containing A and a continuous extension $g : U \to S^1$ of f.

   Hint: first extend f to a map $X \to \mathbf{R}^2$

4. Show that $X \times Y$ is connected if X and Y are.

Part II

1. (a) The Classification Theorem for compact surfaces states that every surface is homeomorphic to exactly one on a certain list.
   What is that list?

   (b) The figure above shows a surface M consisting of a torus T and a Klein bottle K joined together.

      i. Calculate the Euler characteristic of M.

      ii. Identify the surface on the list in your answer to part (a) to which M is homeomorphic.

2. Let S and P denote the 2-sphere and the projective plane respectively. You may assume that $\pi_1(S)$ is the trivial group and that there is a 2-fold covering map $S \to P$.

   (a) Compute $\pi_1(P)$ from the given information.

   (b) Let $X = P_1 \vee P_2$ be the one-point union of two copies of P, and let $x_0 = P_1 \cap P_2$ be the common point of the two projective planes. Use the van Kampen theorem to compute $\pi_1(X)$.

   (c) Draw the universal cover of X and explain your drawing and the covering map in words. What are its deck transformations (also called covering transformations)? Complete proofs are not required.

3. Calculate the fundamental group of the complement in $\mathbf{R}^3$ of a line and a circle disjoint from the line. Note that there are two cases to consider: one where the line goes through the interior of the circle, and the other where it doesn't. Are the spaces obtained in the two situations homotopy equivalent?

4. Let $p : E \to B$ be a covering map, with E path connected. Show that if B is simply-connected, then p is a homeomorphism.

Part III

1. Calculate the homology of the 2-torus T^2 with coefficients in $\mathbf{Z}$, $\mathbf{Z}_2$ and $\mathbf{Z}_3$, respectively.

2. True of False: A continuous map $f : X \to Y$ which induces trivial maps f_* in homology with integer coefficients is nullhomotopic. Explain your answer.

3. Let $f : S^{2n} \to S^{2n}$ be a continuous map. Show that there is a point $x \in S^{2n}$ so that either $f(x) = x$ or $f(x) = -x$.

4. Show that $\mathbf{RP}^3$ and $\mathbf{RP}^2 \vee S^3$ have the same fundamental group and the same homology. Are these spaces homeomorphic? Explain.