TOPOLOGY QUALIFYING EXAM
SEPT. 6, 2007

Instructions. Answer at least two questions from each of the three parts and at least eight questions overall.

PART I

(1) True or False?

(a) Let S^3, as usual, denote the 3-dimensional sphere. S^3 is compact, Hausdorff, and path-connnected.

(b) The continuous image of path-connected space is path-connected.

(c) The continuous image of a compact Hausdorff space is a compact Hausdorff space.

(d) The continuous image of a metrizable space is metrizable.

(e) The 3-sphere is homeomorphic to the one-point compactification of $\mathbb{R}^3$

(f) A covering space mapping is a quotient space mapping.

(g) Let A' denote the transpose of the matrix A and let I_n denote the n-by-n identity matrix. As usual, let $O(n)$ denote the orthogonal group, i.e. the subspace $\{A \in \mathrm{Mat}_{nn}(\mathbb{R}) : AA' = I_n\}$. $O(n)$ is compact.

(h) A continuous bijection is a homeomorphism.

(i) Let $A \subset X$ and $B \subset Y$. Consider $\overline{A} \subset X$ and $\overline{B} \subset Y$ and $\overline{A \times B} \subset X \times Y$. $\overline{A \times B} = \overline{A} \times \overline{B}$.

(2) Let f be a continuous real-valued function defined on a non-empty, compact, connected space. Prove that the image of f is a closed interval.

(3) Choose *one* of the following.
  (a) If $f : X \to \mathbb{R}$ is continuous and X is compact then f attains both its minimum and maximum.
  (b) State and prove Ascoli's Theorem.
  (c) State and prove the Baire Category Theorem.

(4) Let X be a space and R an equivalence relation on X. Let $Y = X/R$ be the set of equivalence classes and $\pi : X \to Y$ the natural onto map. Give Y the quotient topology inherited from X via π.
  (a) Show that a map $f : Y \to Z$ is continuous if and only if $f \circ \pi : X \to Z$ is continuous.
  (b) Show by example that even when X is Hausdorff Y need not be.

(5) Show that $X \times Y$ is connected if X and Y are.

PART II

(1) Describe the Klein bottle and calculate its fundamental group.

(2) Describe, up to **equivalence**, all connected covering spaces of the punctured complex plane, $\mathbb{C} - \{0\}$. Recall that two covering spaces (E_1, π_1, B) and (E_2, π_2, B), having the same base space B are deemed equivalent if and only if there is a homeomorphism $h : E_1 \to E_2$ such that $\pi_1 = \pi_2 \circ h$.

(3) Describe, up to equivalence, the connected covering spaces of $P \times P$ where P is the projective plane. Among these, how many homeomorphism types are there? Explain.

(4) Let X and Y be spaces each homeomorphic to the circle. Let W be the one point union ("wedge product") of X and Y. Let P be the actual product of X and Y. Describe the fundamental groups of W and P. Are they isomorphic? Proof?

(5) Let X be a connected manifold with a finite fundamental group. Show that any continuous function from X to the circle is homotopic to a constant.

PART III

(1) Let $f : S^5 \to S^5$ be defined by

$$f(x_0, x_1, x_2, x_3, x_4, x_5) = (-x_4, x_2, -x_3, -x_1, x_5, x_0).$$

Calculate the degree of f.

(2) Let $X = P_n(\mathbb{C})$, complex projective 2-space and $Y =$ the one-point union of S^2 and S^4. Prove or disprove that for each n, $H_n(X, \mathbb{Z}) \cong H_n(Y, \mathbb{Z})$.

(3) Let X be a CW-complex having exactly 6 cells, one p-cell for each $p \in \{0, 1, 3, 8\}$ and two 5-cells. Describe as completely as you can the homology groups, $H_i(X, Z)$. What can you say about its fundamental group?

(4) Sketch the calculation of $H_p(S^n)$ (all p and n.)

(5) Using what you know about $H_n(P^3(\mathbb{R}), \mathbb{Z})$, all n, calculate $H^n(P^3(\mathbb{R}), \mathbb{Z})$, all n.

(6) Compute the homology groups of $(S^1 \times S^1) \vee S^3$, the wedge product of the torus and a 3-sphere.

(7) Let S_g denote the orientable surface of genus g.
   (a) $H_p(S_g, Z) =$? (all p)
   (b) Sketch the proof.