TOPOLOGY QUALIFYING EXAM
MAY 18, 2007

Instructions. Answer at least two questions from each of the three parts and at least eight questions overall.

PART I

(1) True or False?
   (a) Let S^n denote the n-dimensional sphere. S^n is connected if and only if $n > 0$.
   (b) The continuous image of path-connected space is path-connected.
   (c) The continuous image of a Hausdorff space is Hausdorff.
   (d) The continuous image of a metrizable space is metrizable.
   (e) The 3-sphere is homeomorphic to the one-point compactification of $\mathbb{R}^3$
   (f) Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the map $\pi(x, y) = x$. The map π is a quotient space mapping.
   (g) Let A' denote the transpose of the matrix A and let I_n denote the n-by-n identity matrix. Let $O(n)$ denote the orthogonal group, i.e. the subspace $\{A \in \mathrm{Mat}_{nn}(\mathbb{R}) : AA' = I_n\}$. $O(n)$ is compact.
   (h) Let A' denote the transpose of the matrix A and I_n the n-by-n identity matrix. Let $O(n)$ be the orthogonal group. By definition $O(n) = \{\mathrm{Mat}_{nn}(\mathbb{R}) : AA' = I_n\}$. $O(n)$ is compact.
   (i) Let $A \subset X$ and $B \subset Y$. Let $\overline{A} \subset X$, $\overline{B} \subset Y$, $\overline{A \times B} \subset X \times Y$. $\overline{A \times B} = \overline{A} \times \overline{B}$
   (j) The analogous problem with closure replaced by interior.

(2) State and prove *one* of the following.
   (a) Let f be a continuous, real-valued, function defined on a non-empty, compact, connected space. Prove or disprove: the image of f is a closed interval.
   (b) The Cantor set is nowhere dense.

(3) State and prove *one* of the following.
   (a) Ascoli's Theorem.
   (b) The Baire Category Theorem.

(c) The contraction mapping theorem.

(4) Let X be a metric space containing a countable, dense, subset. Prove that X has a countable basis of open sets.

(5) Let X be a space and R an equivalence relation on X. Let $Y = X/R$ be the set of equivalence classes and $\pi : X \to Y$ the natural onto map. Give Y the quotient topology inherited from X via π.
(a) Show that a map $f : Y \to Z$ is continuous if and only if $f \circ \pi : X \to Z$ is continuous.
(b) Show by example that even when X is Hausdorff Y need not be.

(6) Let X be the product of the indexed family $\{X_\alpha\}_{\alpha \in I}$ of non-empty topological spaces.
(a) Show that X is Hausdorff if and only if each X_α is.
(b) Show that X is path-connected if and only if each X_α is.

PART II

(1) Compute the fundamental group of $P_3(\mathbb{R})$, the 3-dimensional projective space.

(2) Compute the fundamental group of the space obtained from the 2-sphere by identifying the 'north and south poles.'

(3) Describe, up to equivalence, all covering spaces of the punctured complex plane, $\mathbb{C} - \{0\}$.

(4) Describe, up to equivalence, the connected covering spaces of $X \times X$ where X is the projective plane. Among these, how many homeomorphism types are there? Explain.

(5) Let X and Y be spaces each homeomorphic to the circle. Let W be the one point union ($= X \vee Y$, the 'wedge product' of X and Y). Let P be the product, $X \times Y$, of X and Y. Describe the fundamental groups of W and P. Are they isomorphic? Proof?

PART III

(1) Let $X = P_2(\mathbb{C})$, complex projective 2-space and $Y =$ the one-point union of S^2 and S^4. Prove or disprove that for each n, $H_n(X, \mathbb{Z}) \cong H_n(Y, \mathbb{Z})$.

(2) Let $f : S^4 \to S^4$ be defined by

$$f(x_0, x_1, x_2, x_3, x_4) = (-x_4, x_2, -x_3, -x_1, x_0).$$

Calculate the degree of f.

(3) Let X be a CW-complex having exactly 5 cells, one p-cell for each $p \in \{0, 1, 3, 5, 8\}$. Describe as completely as you can the homology groups, $H_i(X, Z)$. What can you say about its fundamental group?

(4) Choose *one* of the following.
   (a) Sketch the calculation of $H_p(S^n)$ (all p and n.)
   (b) *Assuming* relevant facts (*but state them clearly*) about S^n and $H_p(S^n)$, sketch the calculation of the groups $H_p(S^n \times S^m)$ (all p, n and m).

(5) Compute the homology groups of $(S^1 \times S^1) \vee S^3$, the 'wedge product' of the torus and a 3-sphere.

(6) Let S_g denote the orientable surface of genus g.
   (a) $H_p(S_g, Z) =$? (all p)
   (b) Sketch the proof.