Topology Qualifying Exam
Spring 2002

May 21, 2002

Instructions. Do at least eight problems; at least one problem from each part.

Part I

1. State and prove either
   (a) the Baire category theorem; or
   (b) the contraction mapping theorem.

2. Show that the product of connected spaces is connected.

3. (a) Prove that the closed unit interval is a connected topological space.
   (b) Show that the intervals are the connected subsets of the real line. (Here, 'interval' means any set which contains the entire closed interval between any pair of its points.)

4. (a) Show that a compact subspace of a Hausdorff space is closed.
   (b) Is the Hausdorff assumption really necessary? proof?

5. Let X be a compact metric space. Show that any sequence has a convergent subsequence.

6. Let A be a non-empty subset of the metric space X and define $f(x) = \inf\{d(x,a) \mid a \in A\}$. Show that $f(x) = 0$ if and only if $x \in \overline{A}$.

Part II

1. What is the fundamental group of
   (a) S^n for all $n \geq 1$.
   (b) $P_n(\mathbb{R})$ for all $n \geq 1$.
   (c) $\mathbb{R}^n - \{0\}$ for all $n \geq 2$.
   (d) The torus.
   (e) The Klein bottle.

2. Give a detailed statement of and sketch the proof of the following.
   (a) The n-sphere is the universal covering space of $P_n(\mathbb{R})$ for all $n \geq 2$.
   (b) The torus is a 2-sheeted covering space of the Klein bottle.
   (c) The plane is the universal covering space of the Klein bottle

3. Use the Van-Kampen theorem either
   (a) to calculate the fundamental group of the Klein bottle; or
   (b) to prove that, when X is path-connected, $\Sigma(X)$ is simply connected.

4. Let $f : P_2(\mathbb{R}) \to \mathbb{C} - \{0\}$ be a continuous function. Show that f has a continuous square root; i.e. there is a continuous $g : P_2(\mathbb{R}) \to \mathbb{C} - \{0\}$ such that $\forall \in P_2(\mathbb{R}), g(x)^2 = f(x) \in \mathbb{C}$.

5. For sufficiently nice base spaces, B, the category of B's covering spaces is equivalent to the category of G-sets where G is the fundamental group of B.
   (a) Show that, via this equivalence, a covering space of B is path-wise connected if and only if the corresponding G-set has only one G-orbit.
   (b) What is the covering space corresponding to a 2 element G-set on which G acts trivially?

Part III

1. State carefully and sketch the proof of the 'suspension theorem' which relates the homology of X and ΣX.

2. The map $z \to z^2 : \mathbb{C} \longrightarrow \mathbb{C}$ extends to a continuous self-map of the space $\mathbb{C} \cup \{\infty\}$. Thus we have a self map of S^2 since $\mathbb{C} \cup \{\infty\} \approx S^2$. Calculate its degree.

3. Let $X = (X, \{X_n\}_{n \geq 0})$ be a CW complex having exactly 5 cells; one n-cell for each of the dimensions $\{0, 1, 3, 5, 6\}$.

   (a) Say as much as you can about the groups $H_p(X)$.

   (b) Give two examples of such complexes having non-isomorphic homology.

4. Let $A \subset X$.

   (a) Suppose X is path-connected but that A is not. What can you say about $H_1(X, A)$?

   (b) Suppose, further, that X is contractible. Now what?

5. Let A be a retract of X.

   (a) Show that the inclusion map of A in X induces a one-to-one homomorphism in homology.

   (b) By considering the exact sequence for the pair (X, A), show that, for all p

   $$0 \longrightarrow H_p(A) \longrightarrow H_p(X) \longrightarrow H_p(X, A) \longrightarrow 0$$

   is an exact sequence.

   (c) Show that all these short exact sequences are (left-) split and hence that $H_p(X) \cong H_p(A) \oplus H_p(X, A)$.

6. Let X be the oriented surface of genus 2 (i.e. the connected sum of the torus with itself).

   (a) $H_p(X) = ?$

   (b) Justify your answer to a.

7. Consider a CW-complex. Prove that the group of n-dimensional cycles of the cellular chain complex is isomorphic to the n-th homology of the n-skeleton.