Prop^n If $X = \{pt\}$ then $H_n(X) = 0 \quad n > 0$
$H_0(X) \cong \mathbb{Z}$

Proof $C_n(X) \cong \mathbb{Z}$ generated by $\sigma_n : \Delta^n \to \{pt\}$

so $\cdots \to \underset{\mathbb{Z}}{C_{n+1}(X)} \to \underset{\mathbb{Z}}{C_n(X)} \to \underset{\mathbb{Z}}{C_{n-1}(X)} \to \cdots \to \underset{\mathbb{Z}}{C_2(X)} \to \underset{\mathbb{Z}}{C_1(X)} \to 0$

$$\partial \sigma_n = \sum (-1)^n \sigma_n|_{[v_0, \dots \hat{v_i} \dots v_n]} = \sum_{i=0}^{n} (-1)^n \sigma_{n-1} = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$$

$$\xrightarrow[\cong]{} \overset{C_3}{\mathbb{Z}} \xrightarrow[0]{\partial_3} \overset{C_2}{\mathbb{Z}} \xrightarrow[\cong]{\partial_2} \overset{C_1}{\mathbb{Z}} \xrightarrow[0]{\partial_1} \overset{C_0}{\mathbb{Z}} \to 0$$

$\Rightarrow H_n(X) = 0 \quad n \geq 1 \qquad H_0(X) = \mathbb{Z}$. $\square$.

Def^n Reduced homology $\tilde{H}_n(X)$. Replace $\cdots \to C_0(X) \to 0$
with $\cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

$H_n(X) = \tilde{H}_n(X)$ for all $n \geq 1$
$\tilde{H}_0(X) = 0$ if X connected.

Fact $H_1(X) = ab(\pi_1 X)$.

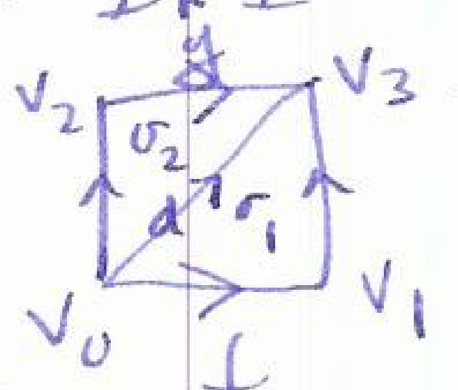
a loop $f : I \to X$ is also a 1-chain.
$f : \Delta^1 \to X$ in fact a 1-cycle
as $\partial f = f|_{[v_1]} - f|_{[v_0]} = 0$.

this gives a map $h : \pi_1(X, x_0) \to H_1(X)$

Th^m $h : \pi_1(X, x_0) \to H_1(X)$ is surjective, and $\ker(h)$ = commutator subgroup, so $ab(\pi_1(X, x_0)) = \pi_1(X, x_0) / \text{commutators} \cong H_1(X)$.

Proof check h is well defined: Notation $f \simeq g$ homotopic $f \sim g$ homologous.

$f \simeq g \Rightarrow \exists$ homotopy $H_t : I \times I \to X$



$H_t = \sigma_1 - \sigma_2$

$c : \Delta^1 \to x_0$ constant path.

$\partial H_t = \overline{f + c - d}^{\,\sigma_1} - \overline{c + g - d}$
$= f - g$

So $f \simeq g \Rightarrow f \sim g$

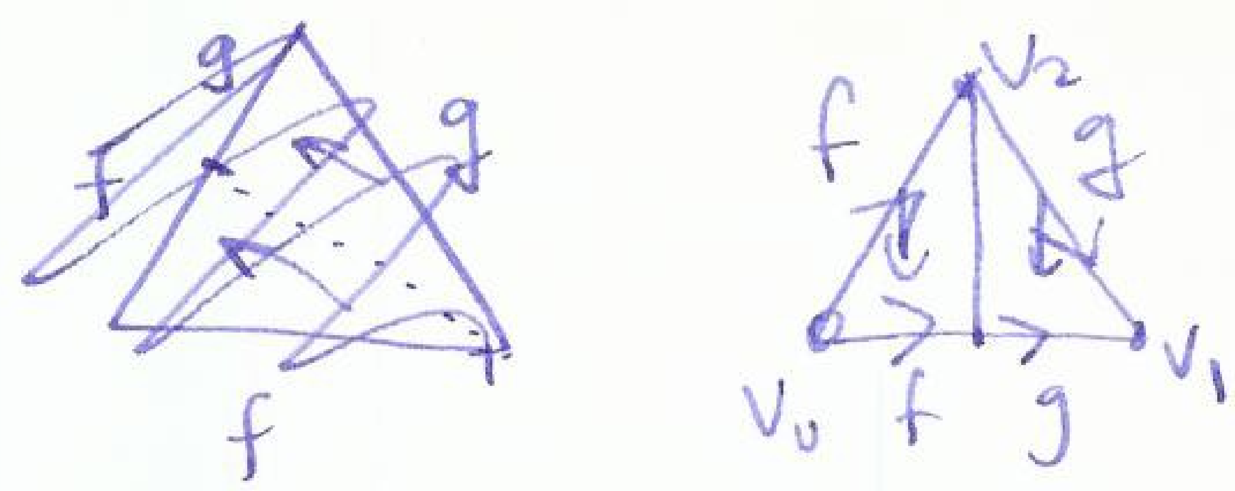
note $c: \Delta^1 \longrightarrow x_0$ constant path $c \sim 0$.

as $c_2: \Delta^2 \longrightarrow x_0$ has boundary $c_2\big|_{[v_0,v_1]} - c_2\big|_{[v_0,v_2]} + c_2\big|_{[v_1,v_2]}$

$$= c - c + c = c$$

so $c \in \mathrm{im}(\partial_2)$ so $c \sim 0$.

check h homomorphism. $f \cdot g \sim f + g$.



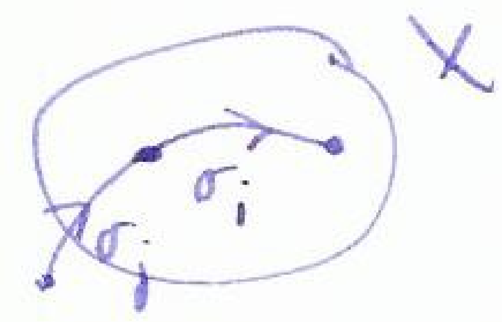
project $\Delta^2 \to$ base $[v_0, v_1]$. $\cancel{\to f\cdot g \to}$ X.

$\partial\sigma = -g + f + f\cdot g \Rightarrow f\cdot g \sim f + g$.

check h surjective. (X path connected)

1 cycle in $C_1(X)$ is $\sum_i n_i \sigma_i$ can relabel so $\sum_i \sigma_i$

spose some $\sigma_i : \Delta^1 \to X$ not a loop



$\partial \sum_i \sigma_i = 0$

$\Rightarrow \exists\, \sigma_j$ s.t. $\sigma_j(v_1) = \sigma_i(v_0)$

now $\sigma_i + \sigma_j \sim \sigma_j \cdot \sigma_i$ so can replace $\sigma_i + \sigma_j$ with $\sigma_j \sigma_i$, so can assume each σ_i is a loop.

X path connected, chose paths γ_i from x_0 to $\sigma_i(v_0)$

then $\sigma_i \sim \sigma_i + \gamma_i - \gamma_i \sim \underbrace{\gamma_i \cdot \sigma_i \cdot \gamma_i^{-1}}_{\text{loop based at } x_0}$

so $\sum \sigma_i = \sum \underbrace{\gamma_i \cdot \sigma_i \cdot \gamma_i^{-1}}$ class in $\gamma_1 \sigma_1 \gamma_1^{-1} \cdots \gamma_n \sigma_n \gamma_n^{-1}$ single loop in $\pi_1(X)$.

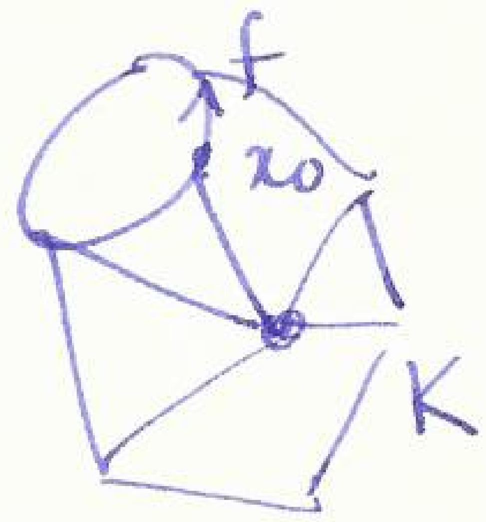
check $\ker(h) = [\pi_1(X), \pi_1(X)]$

$H_1(X)$ abelian $\Rightarrow [\pi_1(X), \pi_1(X)] < \ker(h)$

so we can pair off all of the edges, except one corresponding to ~~$\partial\tau_i$~~. f

we can identify the paired edges to form a 2-complex K, $\sigma : K \to X$

we can homotope σ such that each vertex gets homotoped to x_0.



(i.e. take maximal tree in 1-skeleton)

(homotopy extension property)

(X, A) has the homotopy extension property if every map $X \times \{0\} \cup A \times I \to Y$ can be extended to a map $X \times I \to Y$.

Prop^n If (X, A) is a CW-pair then (X, A) has the homotopy extension property $\square$.

in $\pi_1(X)_{ab}$: $\quad [f] = \sum_{i,j} (-1)^j n_i [\tau_{ij}] = \sum n_i [\partial\sigma_i]$.

$[\partial\sigma_i] = [\tau_{i0} - \tau_{i1} + \tau_{i2}]$. $\quad \sigma_i$ gives a nullhomotopy of $\tau_{i0} - \tau_{i1} + \tau_{i2}$ (as loops!) $\square$

$\Rightarrow [f] = 0$ in $\pi_1(X)_{ab}$.

Fact we can match edges in pairs to form a surface, boundary of surface is a commutator.

Exercise $\quad f = \partial\left(\sum_i n_i \sigma_i\right) \Rightarrow$ surface is orientable.
$\quad n = \pm 1$

Homotopy Invariance

Th^m Homotopy equivalent spaces have isomorphic homology groups.

we'll show $f : X \to Y$ cts induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ for each n, and $f \simeq g$ ~~homotopy equivalent~~ $\Rightarrow f_* = g_*$ as homomorphisms.

$f : X \to Y$ determines $f_\# : C_n(X) \to C_n(Y)$

$(\sigma : \Delta^n \to X) \mapsto f\sigma : \Delta^n \to Y$. extend linear

$\sum n_i \sigma \mapsto \sum n_i f\sigma$

note

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, "f_\#"] & C_n(X) \arrow[r] \arrow[d, "f_\#"] & C_{n-1}(Y) \arrow[r] \arrow[d, "f_\#"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

commutes:

$$f_\# \partial = \partial f_\# .$$

$$f_\# \partial(\sigma) = f_\# \left(\sum_{i=0}^{n} (-1)^n \sigma|_{[v_0 \dots \hat{v}_i \dots v_n]} \right)$$

$$= \sum_{i=0}^{n} (-1) f\sigma|_{[v_0 \dots \hat{v}_i \dots v_n]} = \partial(f_\# \sigma).$$

Def^n $f_\# : C_n(X) \to C_n(Y)$ is a <u>chain map</u> if $\partial f_\# = f_\# \partial$.

claim $f_\#$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$.

check: cycles $\mapsto$ cycles

$\partial \alpha = 0 \Rightarrow \partial f_\# \alpha = 0$.

$$\begin{tikzcd} \beta \arrow[r, mapsto, "\partial\beta ="] & \alpha \arrow[r, mapsto, "\partial"] & \partial\alpha = 0 \\ C_{n+1}(X) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_{n-1}(X) \arrow[d] \\ C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r, "f_\#\alpha"] & C_{n-1}(Y) \\ f_\# \beta \arrow[r, mapsto] & \partial f_\# \beta \arrow[r, mapsto] & \partial f_\# \alpha = f_\# \partial \alpha = f_\#(0) = 0. \end{tikzcd}$$

$$\partial f_\# \beta \;\|\; f_\# \partial \beta = f_\# \alpha.$$

boundaries $\mapsto$ boundaries.

$\partial \beta = \alpha \qquad f_\# \alpha$

$\partial(f_\# \beta) = f_\#(\partial \beta) = f_\#(\alpha)$ so $f_\# \alpha$ is a boundary

<u>Key facts</u> ① $(fg)_* = f_* g_*$ $\qquad X \xrightarrow{f} Y \xrightarrow{g} Z$

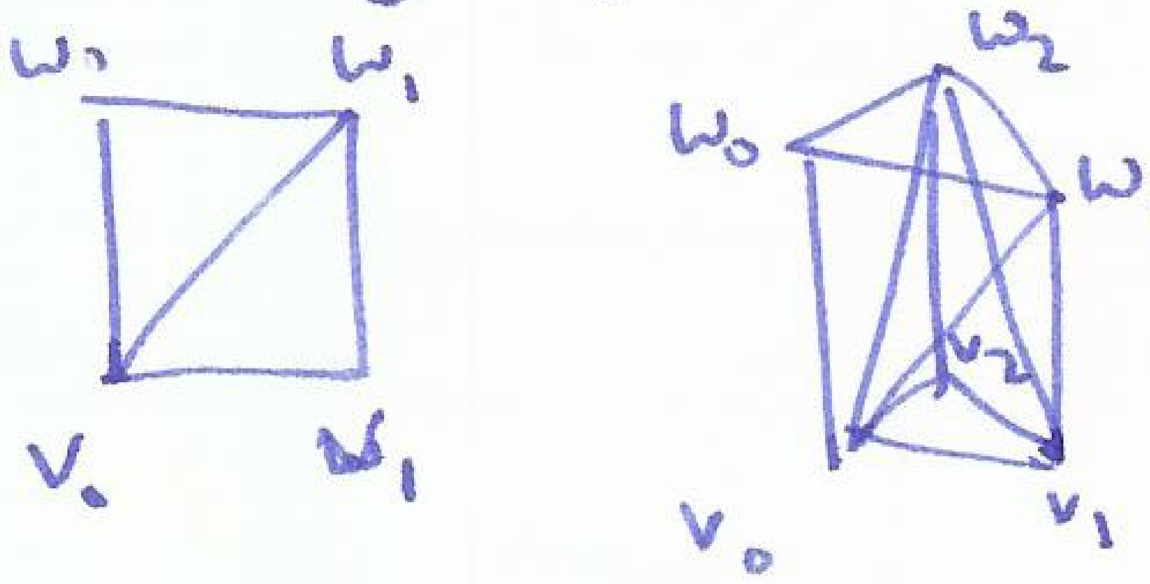
② $\mathbb{1}_* = \mathbb{1}$

Thm If two maps $f, g: X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_*: H_n(X) \to H_n(Y)$.

Corollary $f: X \to Y$ homotopy equivalence $\Rightarrow$ $f_*: H_n(X) \to H_n(Y)$ isomorphism.

i.e. if X contractible then $H_n(X) = 0 \quad n \geq 1$
$H_0(X) \cong \mathbb{Z}$.

Proof key step: can divide $\Delta^n \times I$ into $(n+1)$-simplices



$\Delta^n \times \{1\} = [w_0, w_1, \dots w_n]$

$\Delta^n \times \{0\} = [v_0, v_1, \dots v_n]$

Fact: $(n+1)$-simplices $[v_0, v_1 \dots v_i, w_i, \dots w_n]$ form a simplicial structure on $\Delta^n \times I$

let $F: X \times I \to Y$ be a homotopy from f to g

define $P: C_n(X) \to C_{n+1}(Y)$

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \underbrace{F \circ (\sigma \times \mathbb{1})}_{\Delta^n \times I \to X \times I \to Y.}\Big|_{[v_0, \dots v_i, w_i, \dots w_n]}.$$

claim: $\underbrace{\partial P}_{\text{boundary of } \Delta^n \times I} = \underbrace{g_\#}_{\text{top}} - \underbrace{f_\#}_{\text{bottom}} - \underbrace{P\partial}_{\text{sides}}$

check $$\partial P(\sigma) = \sum_{j\le i} (-1)^i(-1)^j F\circ(\sigma\times\mathbb{1})\big|_{[v_0,\dots \hat{v}_j,\dots v_i, w_i,\dots w_n]}$$
$$+ \sum_{j\ge i} (-1)^i(-1)^{j+1} F\circ(\sigma\times\mathbb{1})\big|_{[v_0,\dots v_i, w_i,\dots \hat{w}_j,\dots w_n]}$$

interior faces $i=j$ cancel out, except for $F\circ(\sigma\times\mathbb{1})\big|_{[\hat{v}_0, w_0,\dots w_n]} = g_\#(\sigma)$.

and $-F\circ(\sigma\times\mathbb{1})\big|_{[v_0,\dots v_n,\hat{w}_n]} = -f_\#(\sigma)$

terms with $i\neq j$ are exactly $-P\partial(\sigma)$ as

$$P\partial(\sigma) = \sum_{i<j} (-1)^i(-1)^j F\circ(\sigma\times\mathbb{1})\big|_{[v_0,\dots v_i, w_i,\dots \hat{w}_j,\dots w_n]}.$$
$$+\sum_{i>j} (-1)^{i-1}(-1)^j F\circ(\sigma\times\mathbb{1})\big|_{[v_0,\dots \hat{v}_j,\dots v_i, w_i,\dots w_n]}.$$

$$\begin{array}{ccccccc}
\cdots\to & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \to\cdots \\
 & f_\# \downarrow\downarrow g_\# & \swarrow^{P} & f_\# \downarrow\downarrow g_\# & \swarrow^{P} & f_\# \downarrow\downarrow g_\# . & \\
\cdots\to & C_{n+1}(Y) & \xrightarrow[\partial]{} & C_n(Y) & \xrightarrow[\partial]{} & C_{n-1}(Y) & \to\cdots
\end{array}$$

$\partial P = g_\# - f_\# - P\partial$

$\partial P + P\partial = g_\# - f_\#$

$\alpha\in C_n(X)$ cycle $\partial\alpha = 0$

$$\downarrow$$

$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\underbrace{\partial(\alpha)}_{=0}$$

$\uparrow$

this is a boundary!

So $g_\#(\alpha) \sim f_\#(\alpha)$ in $H_n(X)$. i.e. $g_*([\alpha]) = f_*([\alpha])$. $\square$

Defn P is a chain homotopy between $f_\#$, $g_\#$ if $g_\# - f_\# = \partial P + P\partial$.