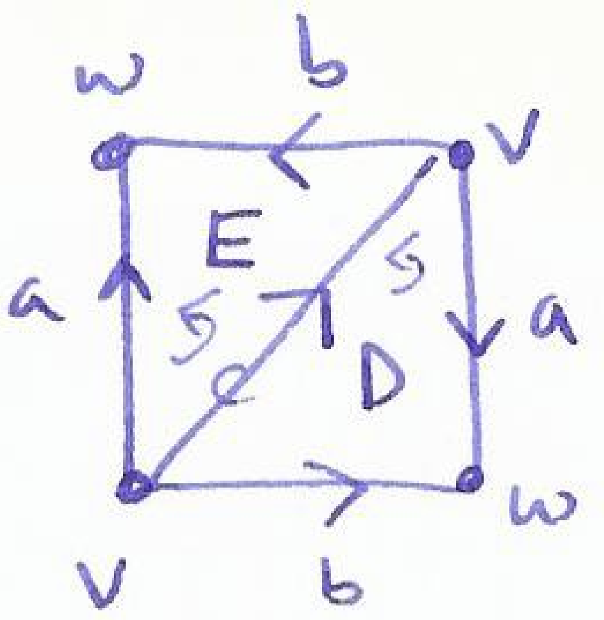
$X = \mathbb{RP}^2$



$$C_3 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z}^2 \to 0$$

$D \mapsto -a+b-c$

$E \mapsto -a+b+c$

$a \mapsto w-v$

$b \mapsto w-v$

$c \mapsto v-v=0$

$H_2(X) = 0 \quad \ker(\partial_2) = 0$

$H_1(X) = \underbrace{\ker(\partial_1)}_{\cong \mathbb{Z}^2} / \operatorname{im}(\partial_2) \cong \mathbb{Z}^2$. $\quad \ker(\partial_1)$ gen by $\{\overset{\alpha}{a-b}, \overset{\beta}{c}\}$.

$\operatorname{im}(\partial_2)$ gen by $\{-a+b, -c, -a+b+c\}$.

change basis to $\{\alpha+\beta, \beta\} \supset \{\alpha+\beta, \alpha+\beta-2\beta\}$. $\quad \{-\alpha-\beta, -\overset{\alpha}{\beta}+\beta\}$.

$\{\alpha, \beta\} \supset \{\alpha, 2\beta\}$. so $\quad \{\alpha+\beta, \alpha-\beta\}$.

$H_1(X) \cong \mathbb{Z}/2\mathbb{Z}$

$H_0(X)$ basis $\{w-v, v\}$

$\cong \mathbb{Z}$.

We've defined Δ-complex homology (could also define simplicial homology, cellular homology?).

problems – does $H_n^\Delta(X)$ depend on Δ-structure? A: No.

– does a map $f: X \to Y$ induce map $f_*: H_*(X) \to H_*(Y)$ A: Yes: simplicial approximation theorem.

Thm If K is a finite simplicial complex, and L is an arbitrary simplicial complex, then any map $f: K \to L$ is homotopic to a map which is simplicial with respect to some (iterated barycentric) subdivision of K.

Fancier version:

Singular homology

Defn A singular n-simplex is a cts map $\sigma: \Delta^n \to X$

Let $C_n(X)$ be the free abelian group with basis consisting of singular n-simplices.

elements are n-chains: $\sum_{i=1}^{n} n_i \sigma_i$.

The boundary map $\partial_n: C_n(X) \to C_{n-1}(X)$ is

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots v_n]}$$

Remark previous argument implies $\partial_n\partial_{n+1} = 0$ ($\partial^2 = 0$)

Defⁿ The singular homology group $H_n(X) = \ker\partial_n / \operatorname{Im}\partial_{n+1}$.

Remark
- homeomorphic spaces clearly have isomorphic $H_n(X)$.
- how do we calculate $H_n(X)$? (e.g. if $X = X^{(n)}$ is $H_{n+k}(X) = 0$?).
- 1-cycles are collection of loops, 2-cycles are maps of surfaces.

Propⁿ If X has path components X_α then $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$

Proof $\sigma(\Delta^n)$ is path connected so lies in a single path component. □.

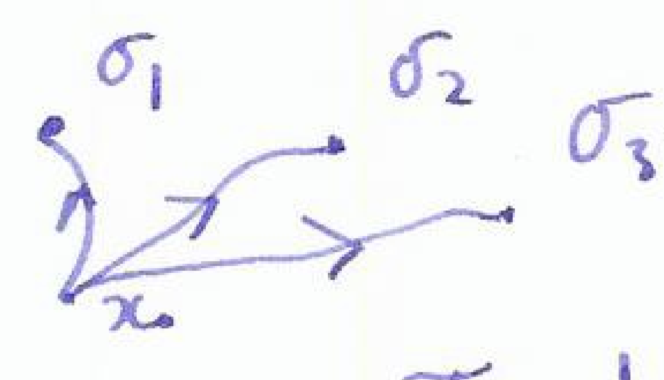
Propⁿ X path connected, $X \neq \emptyset$ then $H_0(X) \cong \mathbb{Z}$

Proof $H_0(X) = C_0(X)/\operatorname{Im}\partial_1$ $\quad \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$

define a homomorphism $\varepsilon : C_0(X) \to \mathbb{Z}$, $\sum_i n_i\sigma_i \mapsto \sum_i n_i$ (surjective! take any fixed $\sigma_1 : \Delta^0 \to X$).

claim $\ker\varepsilon = \operatorname{Im}\partial_1$

spose $\sigma : \underset{[v_0,v_1]}{\Delta^1} \to X$, $\partial_1\sigma = \sigma|_{[v_1]} - \sigma|_{[v_0]}$, so $\varepsilon(\partial_1\sigma) = 1 - 1 = 0$



so $\operatorname{Im}\partial_1 \subset \ker\varepsilon$.

now spose $\sum_i n_i\sigma_i \in \ker\varepsilon$, i.e. $\sum_i n_i = 0$.

let $x_0 \in X$ be a basepoint. As X path connected, can choose path $\tau_i : \Delta^1 \to X$ with $\tau_i(v_0) = x_0$, $\tau_i(v_1) = \sigma_i(pt)$

consider $\partial\left(\sum n_i\tau_i\right) = \sum n_i\partial\tau_i = \sum n_i\left(\tau_i|_{[v_1]} - \tau_i|_{[v_0]}\right)$.

$= \sum n_i\sigma_i - \underbrace{\sum n_i x_0}_{=0} = \sum n_i\sigma_i$. so $\ker\varepsilon \subset \operatorname{im}(\partial_1)$. □.