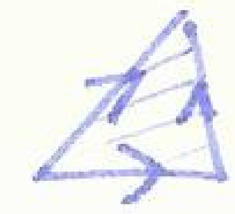
Each edge in a Δ-complex is oriented,  ok  not ok.

face identifications always preserve the orderings of the vertices.

so no two distinct points in the interior of a face may be identified.

so $X = \coprod$ open simplices $e^n_\alpha \subset \Delta^n$. Let $\sigma_\alpha : \Delta^n \to X$

restricts to homeo on e^n_α. Fact: this gives a CW-complex structure to X.

Simplicial Homology

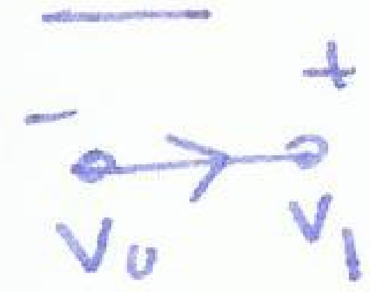
$\Delta_n(X)$ free abelian group with basis open n-simplices e^n_α of X

$\uparrow$ elements are $\sum_\alpha n_\alpha e^n_\alpha$, $n_\alpha \in \mathbb{Z}$ (or $\sum_\alpha n_\alpha \sigma_\alpha$)

formal sums

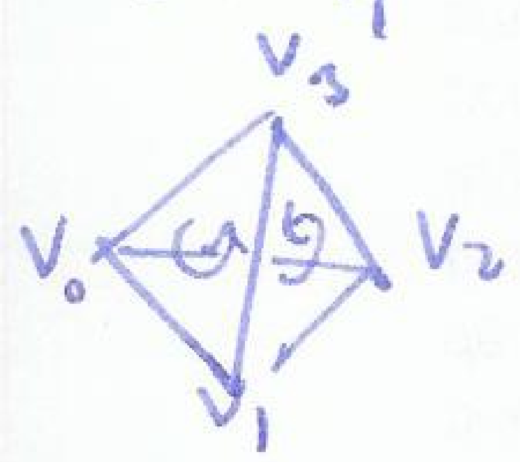
called chains.

Note the boundary of an n-simplex is an $(n-1)$ chain.



$$\partial[v_0, v_1] = [v_1] - [v_0]$$

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$



$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

In general: $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha \big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}.$$

Lemma $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ is zero.

Proof $\partial_n(\sigma) = \sum_i (-1)^i \sigma \big|_{[v_0, \dots \hat{v}_i \dots v_n]}$

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i} (-1)^i (-1)^j \sigma \big|_{[v_0, \dots \hat{v}_j, \dots, \hat{v}_i \dots v_n]}$$

$$+ \sum_{j>i} (-1)^i (-1)^{j-1} \sigma \big|_{[v_0, \dots \hat{v}_i \dots \hat{v}_j \dots v_n]} = 0 \ \square$$

algebraic setup:

$$\to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

a sequence of abelian groups, with homomorphisms between them such that $\partial_n \circ \partial_{n+1} = 0$ called a <u>chain complex</u>

$\partial_n \partial_{n+1} = 0 \implies \operatorname{im} \partial_{n+1} \subset \ker \partial_n$

define the n-th homology group of the chain complex to be

$H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}$ elements of $\ker \partial_n$ are <u>cycles</u>
elements of $\operatorname{im} \partial_n$ are <u>boundaries</u>

elements of H_n are cosets of $\operatorname{im} \partial_{n+1}$ called homology classes.

two equivalent cycles c_1, c_2 are called <u>homologous</u>. i.e. $c_1 - c_2 \in \operatorname{im} \partial_n$.

$C_n = \Delta_n(X)$ $H_n^{\Delta}(X)$ is the n-th <u>simplicial homology group</u>

<u>Examples</u> $X = \{pt\}$

$$C_2 \to C_1 \to C_0 \to 0$$
$$0 \to 0 \to \mathbb{Z} \to 0$$

$H_n(X) = 0$ $n > 0$
$H_0(X) \cong \mathbb{Z}$.

$X =$ 

$[v_0, v_1] = e$

$$C_2 \to C_1 \to C_0 \to 0$$
$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to 0$$
$$[v_0, v_1] \xmapsto{\partial_1} [v_1] - [v_0]$$

$\ker(\partial_1) = 0$ so $H_n(X) = 0$ $n > 0$
$H_0(X) \cong \mathbb{Z}$

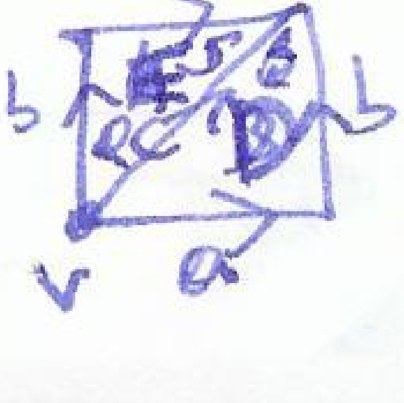
as $\mathbb{Z}^2$ has basis $\{[v_1] - [v_0], [v_0]\}$.

$X = S^1 =$ 

$$C_2 \to C_1 \to C_0 \to 0$$
$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0$$
$$e \xmapsto{\partial_1} [v] - [v] = 0.$$

so $H_n(S^1) = 0$ $n > 2$
$H_1(S^1) \cong \mathbb{Z}$
$H_0(S^1) \cong \mathbb{Z}$

$$C_4 \to C_3 \to C_2 \to C_1 \to C_0 \to 0$$
$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0.$$

$X =$ 

$D \mapsto a + b - c$
$E \mapsto c - a - b + c$
$a \mapsto 0$
$b \mapsto 0$
$c \mapsto 0$.

$\partial D = -\partial E$

$H_0(X) \cong \mathbb{Z}$
$H_1(X) \cong \mathbb{Z}^2$
$H_2(X) \cong \mathbb{Z}$.
$H_n(X) \cong 0$ $n \geq 3$