claim EG contractible: do linear homotopy $x \in [g_0, \dots g_n]$ to $[e]$ in $[e, g_0, g_1, \dots, g_n]$. This is well defined, as if we restrict to a face, i.e. $x \in [g_0, \dots \hat{g_i}, \dots g_n]$ then get linear homotopy in $[e, g_0, \dots \hat{g_i} \dots g_n]$

(note not a deformation retraction as $[e]$ goes to $[e]$ along $[e, e]$.

$G \curvearrowright EG \qquad g[g_0, \dots g_n] = [gg_0, \dots, gg_n]$

claim: action is a covering space action.

so the quotient map $EG \to EG/G$ is the universal cover of $BG = EG/G$

and BG is a $K(G,1)$ $\square$.

Remark BG has a single vertex $[[e]]$ but is infinite dimensional.

~~simplices may be written as: $[g_0,$~~

Fact If G contains torsion then $K(G,1)$ is infinite dimensional.

Thm The homotopy type of a CW complex $K(G,1)$ is uniquely determined by G. $\square$

Graphs of groups

Γ graph, connected, oriented $\qquad$ vertex $v \leftrightarrow$ group G_v

edge $e \leftrightarrow$ homomorphisms ϕ_e

now build a space: vertex $v \leftrightarrow K(G_v, 1)$.

edge $e \leftrightarrow$ mapping cylinder $B\phi_e$

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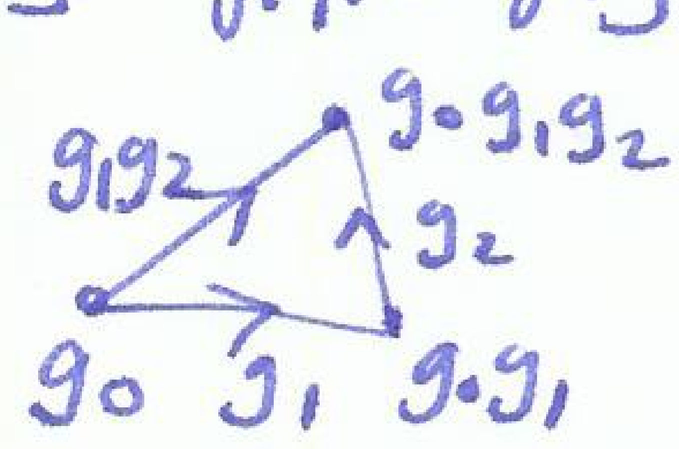
notation $[g_1 | g_2 \cdots | g_n] = [e, g_1, g_1g_2, g_1g_2g_3, \dots g_1g_2 \cdots g_n]$.

simplex in BG

so any simplex $[h_0, h_1, \dots h_n] = h_0 [e, h_0^{-1}h_1, h_0^{-1}h_2, \dots, h_0^{-1}h_n]$.

$= h_0 [h_0^{-1}h_1 | h_1^{-1}h_2 | \dots]$.



functorial: $f: G \to H$ homomorphism

gives $Bf: BG \to BH \quad [g_1 | g_2 | \dots | g_n] \mapsto [f(g_1) | f(g_2) | \dots f(g_n)]$.

§2 Homology

<u>motivation</u> want higher dim invariants. eg. $\pi_1(S^n) = 0$ $n \geq 2$

why not $\pi_k(X, x_0)$? ~~π_k~~ $(I^k, \partial I^k) \to (X, x_0)$.
homotopy classes of maps.

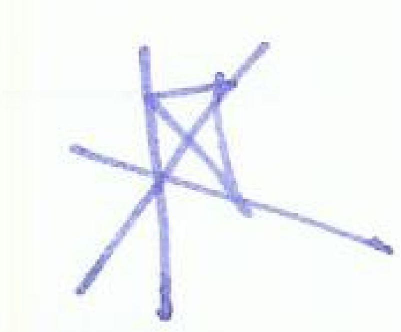
<u>Fact</u> $\pi_k(X, x_0)$ abelian for $k \geq 2$, very difficult to compute.

$\pi_n(S^n) \cong \mathbb{Z}$ $\quad$ $\pi_3(S^2) \cong \mathbb{Z}$. Hopf fibration.  $\mathbb{C}$-lines in $\mathbb{C}^2$.

§2.1 Simplicial and singular homology

<u>Δ-complexes</u>

n-simplex: smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, \dots v_n$ that do not lie in a plane of dim $< n$, equivalently, the vectors ~~v_0~~ $v_1 - v_0, v_2 - v_0, \dots v_n - v_0$ are linearly independent.



v_i vertices of simplex $[v_0, \dots v_n]$

standard n-simplex $v_i =$ unit coordinate basis vectors $\quad \Delta^n = \{(t_0, \dots t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, \ t_i \geq 0\}$

we wish to keep track of the order of the vertices.

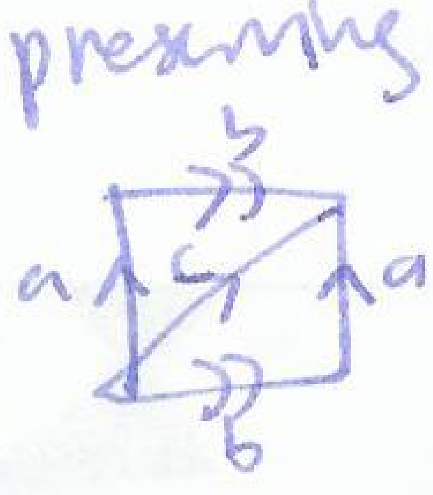
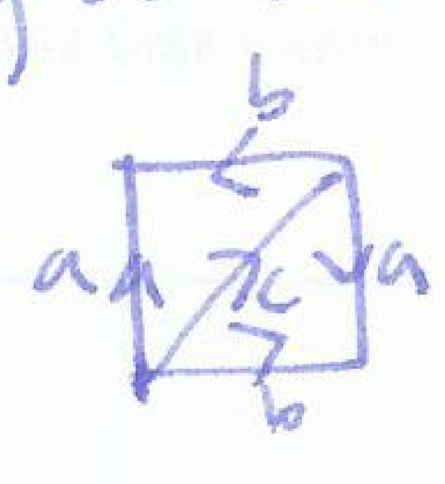
so $[v_0, \dots v_n]$ is the <u>ordered</u> set of vertices.

this orders every sub-simplex, eg edge $[v_i, v_j]$

gives canonical <u>barycentric coordinates</u> on any simplex $\sum t_i v_i$ $\quad t_i \geq 0$, $\sum t_i = 1$.

A <u>face</u> of a simplex ~~ϕ~~ $[v_0, \dots v_n]$ is any (not-necessarily) proper subset of $[v_0 \dots v_n]$.

<u>Def^n</u> A <u>Δ-complex</u> is the quotient space of a collection of disjoint simplices obtained by identifying certain of their faces by the canonical linear maps preserving the ordering of the vertices.

<u>Examples</u> T^2  $\mathbb{R}P^2$  K: