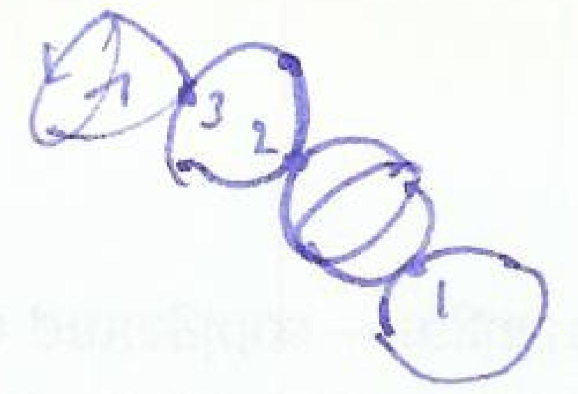
$a \mapsto (1)(2)(3)$
$b \mapsto (123)$

$G \cong \mathbb{Z}/3\mathbb{Z}$

$a \mapsto (12)(3)$
$b \mapsto (1)(23)$



G trivial.

$a \mapsto (12)(3)$
$b \mapsto (123)$

G trivial

Deck transformations and group actions

$p: \tilde{X} \to X$ covering space. The isomorphism $\tilde{X} \to \tilde{X}$ are called the deck transformations or covering transformations. These form a group $G(\tilde{X})$ under composition.

Example

$\mathbb{R}$
$\downarrow$
S^1

deck transformation $x \mapsto x+1$

$G(\mathbb{R} \to S^1) \cong \mathbb{Z}$

$z \mapsto z^n$
$\downarrow p$

deck transformations $G(z \mapsto z^n) \cong \mathbb{Z}/n\mathbb{Z}$.

By the unique lifting lemma a deck transformation is determined by where it sends $\tilde{x}_0$ basepoint in $\tilde{X}$, so if it fixes a point it is the identity.

Def^n A covering space is normal or regular if for every $\tilde{x}_0, \tilde{x}_0'$ in $p^{-1}(x_0)$ there is a deck transformation taking $\tilde{x}_0$ to $\tilde{x}_0'$.

Prop^n Let $p: (\tilde{X}_0, \tilde{x}_0) \to (X, x_0)$ be a covering space and let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Then

a) $\tilde{X} \to X$ is normal iff $H < \pi_1(X, x_0)$ is a normal subgroup.

b) $G(\tilde{X}) \cong N(H)/H$ $\quad N(H) =$ normalizer of H.

In particular if $\tilde{X}$ is regular, $G(\tilde{X}) \cong \pi_1(X, x_0)/H$

if $\tilde{X}$ universal cov' $G(\tilde{X}) \cong \pi_1(X, x_0)$.

Proof recall: change of basepoint $\tilde{x}_0$ to $\tilde{x}_1$ in $p^{-1}(x_0)$ corresponds to conjugation of H in $\pi_1(X, x_0)$ by $[\gamma]$ where $\tilde{\gamma}_{x_0}(1) = \tilde{x}_1$.

$$\text{so}\quad [\gamma] \in N(H) \iff \underset{p_*(\pi_1(\tilde{X}, x_1))}{\underset{\parallel}{[\gamma] H [\gamma]^{-1}}} = \underset{p_*(\pi_1(\tilde{X}, x_0))}{\underset{\parallel}{H}}$$

lifting criterion $\Rightarrow$ $\exists$ deck transformation taking $\tilde{x}_0$ to $\underset{\tilde{\gamma}_{x_0}(1).}{\overset{}{\tilde{x}_1}}$

$$\begin{tikzcd} (\tilde{X}, \tilde{x}_0) \arrow[r, shift left, "\tilde{p}_0"] \arrow[dr, "p_0"'] & (\tilde{X}, \tilde{x}_1) \arrow[l, shift left, "\tilde{p}_1"] \arrow[d, "p_1"] \\ & X \end{tikzcd}$$

i.e. $[\gamma] \in N(H) \iff \exists$ deck transformation $\tau : \tilde{x}_0 \mapsto \tilde{x}_1$.

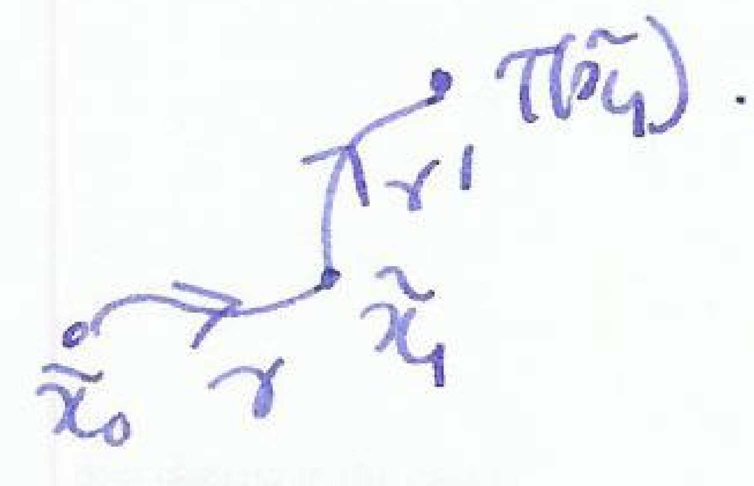
i.e. $\tilde{X} \to X$ regular iff $N(H) = \pi_1(X, x_0)$.

quotient: define $\phi : N(H) \to G(\tilde{X})$

$[\gamma] \longmapsto \tau_{[\gamma]} : \tilde{x}_0 \mapsto \tilde{x}_1$

$\uparrow$ deck transformation.

ϕ is a homomorphism. $\quad \phi' \phi = \tau_{[\gamma']} \tau_{[\gamma]} = \tau_{[\gamma\gamma']}$.



ϕ is surjective by above.

ϕ is injective: kernel $=$ loops in X lifting to loops in $\tilde{X}$ $= p_*(\pi_1(\tilde{X}, \tilde{x})) = H$, so $G(\tilde{X}) \cong N(H)/H$ $\square$.

Cayley graphs / Cayley complexes

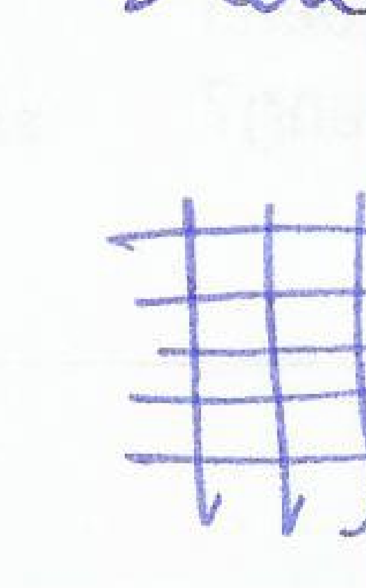
G group, presentation $\langle g_\alpha \mid r_\beta \rangle \rightsquigarrow$ 2-dim cell complex X_G

Cayley graph: vertices $\leftrightarrow$ group elements

(connected) edges $\leftrightarrow$ generators, connect V_g to $V_{g g_\alpha}$.

Cayley complex: glue on 2-cell for each r_β, starting at any vertex.

Example $\langle a, b \mid ab = ba \rangle$.



Fact: Cayley complex $= \tilde{X}$ universal cover of X_G.