$$\begin{array}{ccc} \tilde X_0, \tilde x_0 & \xrightarrow{1_{\tilde X}} & (\tilde X_0, \tilde x_1) \\ \downarrow p_0 & & \downarrow p_0 \\ X, x_0 & & X, x_0 \end{array}$$

$\tilde h$ $\quad$ $\tilde\gamma$ $\quad$ $\tilde h$ $\quad$ $\tilde\gamma \tilde h \bar{\tilde\gamma}$

$1_{\tilde X}$ is a homeomorphism (not basepoint preserving)

ghg^{-1} $\quad$ so $p_*(X_0, \tilde x_1) = H_1$

but then $(\tilde X_0, \tilde x_1)$ is the unique cover with $p_*(\tilde X, \tilde x_1) = H_1$ □.

The simply connected cover is called the universal cover.

$$\begin{array}{ccc} & & \tilde X_1 \\ & \nearrow^{\tilde p} & \downarrow p_1 \\ \tilde X & \xrightarrow[p]{} & X \end{array}$$

lifting criterion implies $\tilde p : \tilde X \to X_1$

s.t. $\quad p = p_1 \tilde p$

so $\tilde p : \tilde X \to \tilde X_1$ is a covering space map.

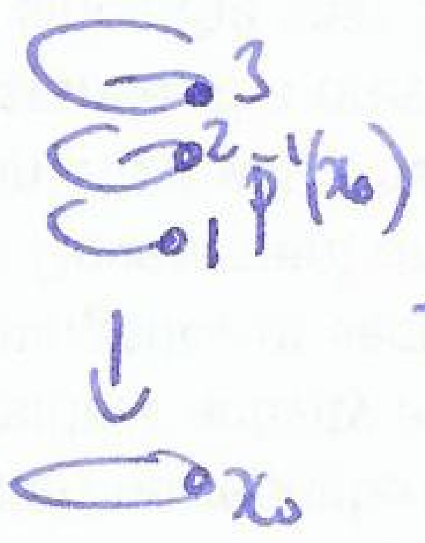
only 1 trivial subgroup, so any two simply connected covers are isomorphic.

partial order on subgroups by inclusion $\leftrightarrow$ partial order on covers by which ones cover each other.

Representing covering spaces by permutations

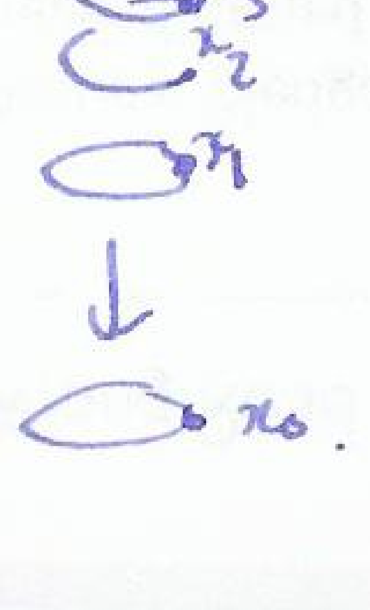
disconnected covering spaces $\leftrightarrow$ permutations
connected covering spaces $\leftrightarrow$ transitive permutations. } actions of $\pi_1(X, x_0)$ on a discrete set i.e. $\rho : \pi_1(X, x_0) \to S_n$.

Example (S^1)



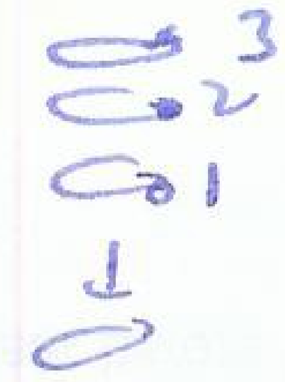
The set is always $p^{-1}(x_0)$

$\pi_1(X) \cong \mathbb{Z} \curvearrowright \{1,2,3\}$ by $\quad 1 \in \mathbb{Z} \to (123)$



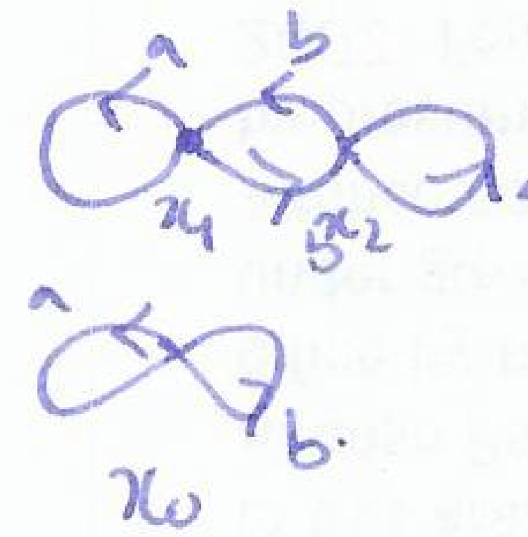
$\pi_1(X) \cong \mathbb{Z} \curvearrowright \{1,2,3\}$ by $(1)(23)$

in fact *any* permutation of $\{1,2,3\}$ gives a 3-fold cover

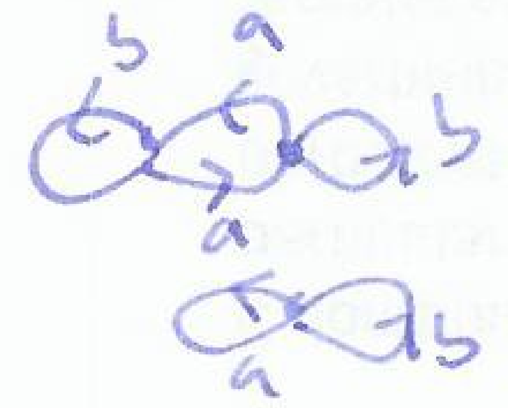


$(1)(2)(3)$.

Example F_2

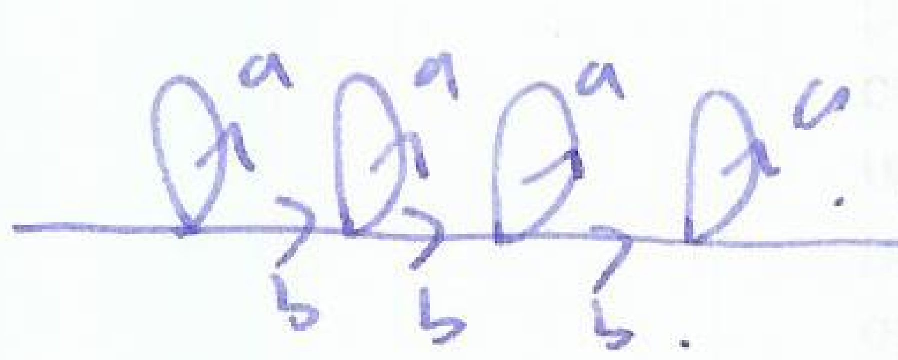


$$F_2 \to S_2$$
$$a \mapsto (1)(2)$$
$$b \mapsto (12)$$



$$F_2 \to S_2$$
$$a \mapsto (12)$$
$$b \mapsto (1)(2).$$

Exercise draw
$$a \mapsto (12)$$
$$b \mapsto (12)$$



$$F_2 \to \mathrm{Sym}(\mathbb{Z})$$
$$a \mapsto \mathbb{1}_{\mathbb{Z}}.$$
$$b \mapsto (n \mapsto n+1)$$

Universal cover: $\pi^{-1}(x_0) \Longleftrightarrow \pi_1(X, x_0)$. — set bijection. action of $\pi_1(X,x_0)$ on itself on the left.

In general: A cover $p: \tilde{X} \to X$ gives a permutation rep^n of $\pi_1(X,x)$ on $p^{-1}(x_0)$ by $[\gamma] \mapsto \tilde{\gamma}(1)$, well defined by homotopy lifting property.

~~Converse: spose $\pi_1(X, x_0) \curvearrowright$ on $p^{-1}(x_0) = F$ want to construct $\tilde{X}$.~~
~~$\rho: \pi_1(X,x_0) \to \mathrm{Sym}(F)$~~

~~Let $\tilde{X}_0$ be the universal cover, and consider $h: \tilde{X}_0 \times F \to \tilde{X}$~~
~~$[\gamma], x_0 \mapsto$ lift of γ starting at $\tilde{x}_0$.~~

~~claim: this is a covering space ($h^{-1}(U)$ discrete copies of U) D.~~

~~in general h not injective, but induces quot equivalence relation on~~

~~put equivalence relation on $\tilde{X}_0 \times F$ by $[\gamma] \sim [\gamma']$ if~~

This is called the action of $\pi_1(X, x_0)$ on the fiber $F = p^{-1}(x_0)$.

notation: $\gamma \longmapsto L_\gamma \in \mathrm{Sym}(F)$

we can reconstruct $\tilde{X}$ from action of $\pi_1(X, x_0)$ on $p^{-1}(x_0) = F$

let $\tilde{X}_1$ be the universal cover, and define $h: \tilde{X}_1 \times F \longrightarrow \tilde{X}$

$$([\gamma], \tilde{x}_0) \longmapsto \tilde{\gamma}_{\tilde{x}_0}(1)$$

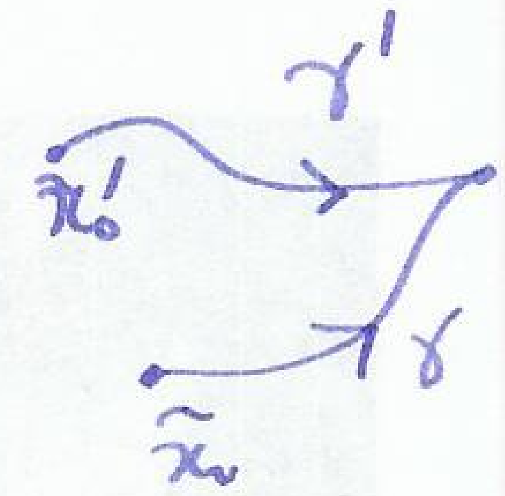
where $\tilde{\gamma}_{\tilde{x}_0}(1)$ is lift of γ starting at $\tilde{x}_0 \in F$.

note: h is cts as open sets in $\tilde{X}_1 \times F$ look like $U_{[\gamma]} \times \{\tilde{x}_0\}$ so in fact local homeo

h is surjective as X path connected

so can take quotient $\tilde{X}_1 \times F/\sim$ where $([\gamma], \tilde{x}_0) \sim ([\gamma'], \tilde{x}_0')$ if $h(\ \text{"}\ ) = h(\ \text{"}\ )$.

spose $h(([\gamma], \tilde{x}_0)) = h(([\gamma'], \tilde{x}_0'))$, then:



so $\tilde{x}_0' = L_{\gamma\bar{\gamma}'}(\tilde{x}_0)$

let λ = loop $\gamma\bar{\gamma}'$ in X, then $h(([\gamma], \tilde{x}_0)) = ([\lambda\gamma], L_\lambda(\tilde{x}_0))$

and this works for any loop λ, so we get a map $\tilde{X}_1 \times F/\sim \longrightarrow \tilde{X}$

note: this map is a bijection, and so homeo as h local homeo, so $\tilde{X}$ depends only on L.

Corollary n-sheeted covers of X $\longleftrightarrow$ representations $\pi_1(\tilde{X}, x_0) \to Sym(n)$ (up to conjugacy)

Example: construct all connected 3-fold covers of $\mathbb{RP}^2 \vee S^1$

$\pi_1(\mathbb{RP}^2 \vee S^1) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z} = \langle a, b \mid a^2 \rangle \longrightarrow Sym(3)$

$a \mapsto (1)(2)(3)$ or $a \mapsto (12)(3)$

$b \mapsto (123)$ $\quad$ $b \mapsto (1)(23)$

only transitive choice! $\quad$ $b \mapsto (123)$