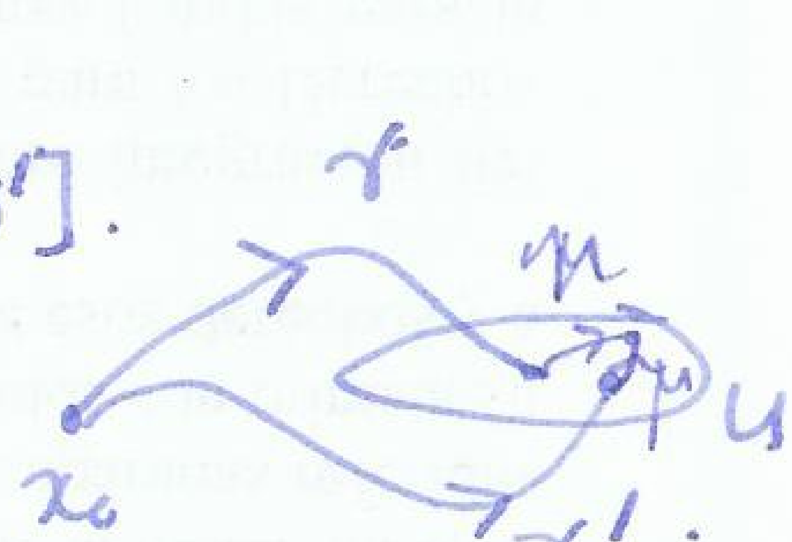
set $\tilde{X}_H = \tilde{X}/\sim$. <u>Claim</u> $\tilde{X}_H$ is a covering space.

$p: \tilde{X}_H \to X$ let $U \subset X$ consider $p^{-1}(U) = \coprod_{\gamma \in \pi_1(X)} U_{[\gamma]} / \sim$.

$[\gamma] \mapsto \gamma(1)$

but if $[\gamma\eta] \in U_{[\gamma]}$ identified with $[\gamma'\mu] \in U_{[\gamma']}$.

i.e. $[\gamma\eta\bar{\mu}\bar{\gamma}'] \in H$.



but now for any $[\gamma\nu] \in U_{[\gamma]}$ $[\gamma\nu\bar{\nu}\eta\bar{\mu}\bar{\gamma}'] \in H$

$$\underbrace{[\gamma\nu]}_{\in U_{[\gamma]}}\underbrace{[\gamma'\mu\bar{\eta}\nu]}_{\in U_{[\gamma']}}$$ so $U_{[\gamma]} = U_{[\gamma']}$.

<u>claim</u>: $p_*(\pi_1(\tilde{X}_H, \tilde{x})) = H$.

↑ loops which lift to loops.

$\gamma$ loop in $X$, then $\tilde{\gamma}_t = [\gamma_t]$ so if $\gamma$ lifts to a loop

then $[\gamma_1] \sim [x_0]$, i.e. $[\gamma_1 \cdot c] \in H \Rightarrow [\gamma_1] \in H$. as required $\square$.

<u>Recall</u>: two covering spaces are isomorphic if there is a homeo $f: \tilde{X}_1 \to \tilde{X}_2$ s.t. $p_1 = p_2 f$

<u>Prop$^n$</u> $X$ path connected locally path connected, then two covering spaces $p_1: \tilde{X}_1 \to X$ and $p_2: \tilde{X}_2 \to X$ are isomorphic by $f: \tilde{X}_1 \to \tilde{X}_2$ taking the basepoint $\tilde{x}_1 \in p^{-1}(x_0)$ to $\tilde{x}_2 \in p^{-1}(x_0)$ iff

$p_*(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_*(\pi_1(\tilde{X}_2, \tilde{x}_2))$ equal, <u>not</u> isomorphic!

**Proof**: $\Rightarrow$ spose there is an isomorphism $f:(\tilde X_1,\tilde x_1)\to(\tilde X_2,\tilde x_2)$

$$\begin{tikzcd} \tilde X_1 \arrow[r, "f"] \arrow[dr, "p_1"'] & \tilde X_2 \arrow[d, "p_2"] \\ & X \end{tikzcd}$$

then $p_1 = p_2 f$ and $p_2 = p_1 f^{-1}$ $\Rightarrow$ $p_{1*}(\pi_1(\tilde X_1,\tilde x_1)) = p_{2*}(\pi_1(\tilde X_2,\tilde x_2))$

$\Leftarrow$ Suppose $p_{1*}(\pi_1(\tilde X_1,x_1)) = p_{2*}(\pi_1(\tilde X_2,\tilde x_2))$

$$\begin{tikzcd} \tilde X_1 \arrow[r, shift left, "\tilde p_1"] \arrow[dr, "p_1"'] & \tilde X_2 \arrow[l, shift left, "\tilde p_2"] \arrow[d, "p_2"] \\ & X \end{tikzcd}$$

lifting criterion: $p_1$ lifts to a map $\tilde p_1:\tilde X_1\to\tilde X_2$ with $p_1 = p_2\tilde p_1$

similarly $p_2$ lifts to $\tilde p_2$ with $p_2 = p_1\tilde p_2$

now consider

$$\begin{tikzcd} \tilde X_1 \arrow[r, "\tilde p_1"] \arrow[rr, bend left, "1_{\tilde X_1}"] \arrow[dr, "p_1"'] & \tilde X_2 \arrow[r, "\tilde p_2"] & \tilde X_1 \arrow[dl, "p_1"] \\ & X_1 & \end{tikzcd}$$

$p_1 1_{X_1} = p_1$

$p_1\tilde p_2\tilde p_1 = p_2\tilde p_1 = p_1$

so both $1_{X_1}$ and $\tilde p_2\tilde p_1$ cover $p_1$. But $1_{X_1}(\tilde x_1) = \tilde x_1$

$\tilde p_2\tilde p_1(\tilde x_1) = \tilde x_1$

so by uniqueness of lifts $\tilde p_2\tilde p_1 = 1_{X_1}$ similarly $\tilde p_1\tilde p_2 = 1_{X_2}$.

so $\tilde p_1 = \tilde p_2^{-1}$ homeomorphisms. $\square$

Final bit: changing base point $\Leftrightarrow$ conjugation in $\pi_1(X,x_0)$.


**Proof** $\begin{array}{c}\tilde X,\tilde x_0\\ \downarrow p\\ X,x_0.\end{array}$ Let $\tilde x_1$ be a different basepoint in $p^{-1}(x_0)$


let $\tilde\gamma$ be a path from $\tilde x_0$ to $\tilde x_1$, and let $\gamma = p\tilde\gamma$ loop in $(X,x_0)$

let $H_0 = p_*(\pi_1(\tilde X,\tilde x_0))$. recall $\beta[\tilde\gamma]: \pi_1(\tilde X,\tilde x_0)\to\pi_1(\tilde X,\tilde x_1)$ isomorphism.

$H_1 = p_*(\pi_1(\tilde X,\tilde x_1))$

$\beta[\gamma]: H_0 \to H_1$. conjugation

$h \mapsto [\gamma]^{-1}h[\gamma]$

conversely: spose $H_0, H_1$ conjugate in $\pi_1(X,x_0)$.

by $g$, choose loop $[\gamma] = g$ $\quad h\mapsto g^{-1}hg$

~~lifts to $\tilde\gamma . h\tilde\gamma$ loop based at $\tilde x_0$.~~