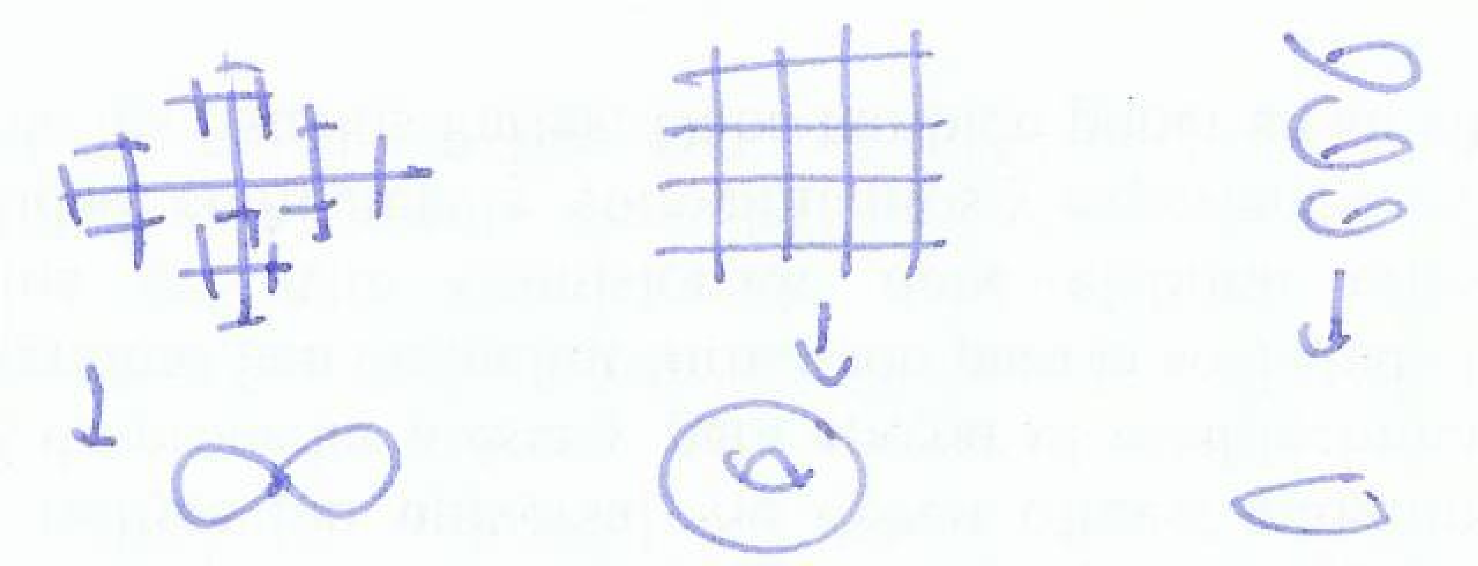
Construction of the universal cover $p: \tilde{X} \to X$ s.t. $p_* \pi_1(\tilde{X})$ trivial.

Examples



X path connected
locally path connected
semi-locally simply connected

Def^n $\tilde{X} = \{ [\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0 \}$

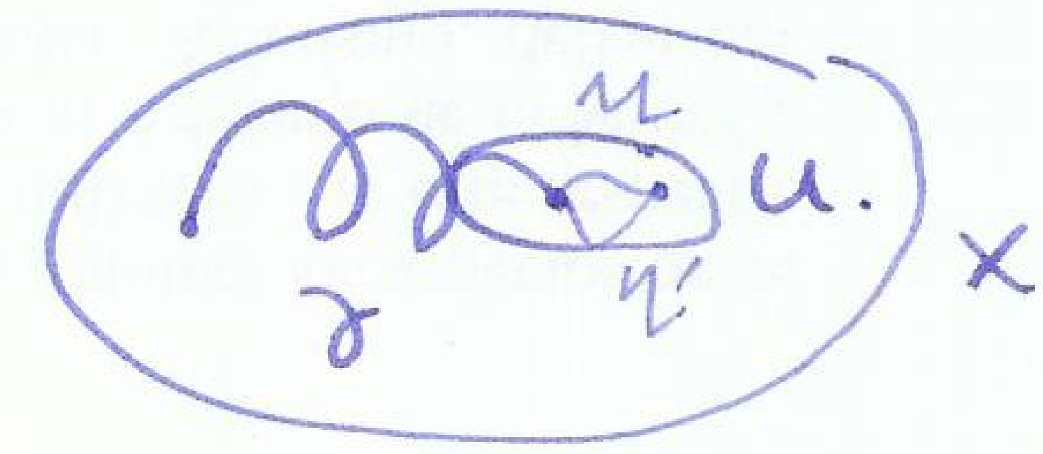
Remark define $p: \tilde{X} \to X$
$[\gamma] \mapsto \gamma(1)$.

X path connected $\Rightarrow$ p surjective.

need a topology on $\tilde{X}$: key point: Let $\mathcal{U}$ be the collection of path connected open sets $U \subset X$ s.t. $\pi_1 U \to \pi_1 X$ is trivial. Then $\mathcal{U}$ is a basis for the topology on X if X is locally path connected and semi-locally simply connected.

Let $U_{[\gamma]} = \{ [\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1) \}$

$p: U_{[\gamma]} \to U$ • surjective as U path connected
• injective as $\pi_1(U) \to \pi_1(X)$ trivial.



• (*) $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$. If $\gamma' = \gamma\eta$ then

$U_{[\gamma']} = \{ [\gamma \cdot \eta \cdot \mu] \mid \mu \text{ is a path in } U \text{ with } \mu(0) = \gamma \cdot \eta(1) \} \subset U_{[\gamma]}$.

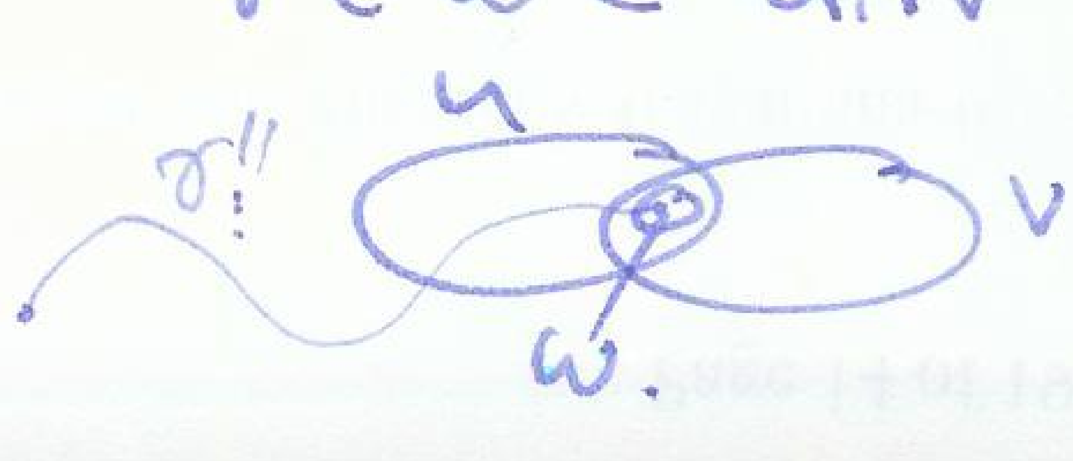
$U_{[\gamma]} = \{ [\gamma \cdot \eta \bar{\eta} \mu] \mid \mu \text{ is a path in } U \text{ with } \mu(0) = \gamma(1) \} \subset U_{[\gamma']}$.

Claim $U_{[\gamma]}$ form a basis for a topology on $\tilde{X}$.

check intersections: Let $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']} = U_{[\gamma'']} \cap V_{[\gamma'']}$

$\mathcal{U}$ basis for top of X, so choose $W \in \mathcal{U}$ with $\gamma'' \in W \subset U \cap V$

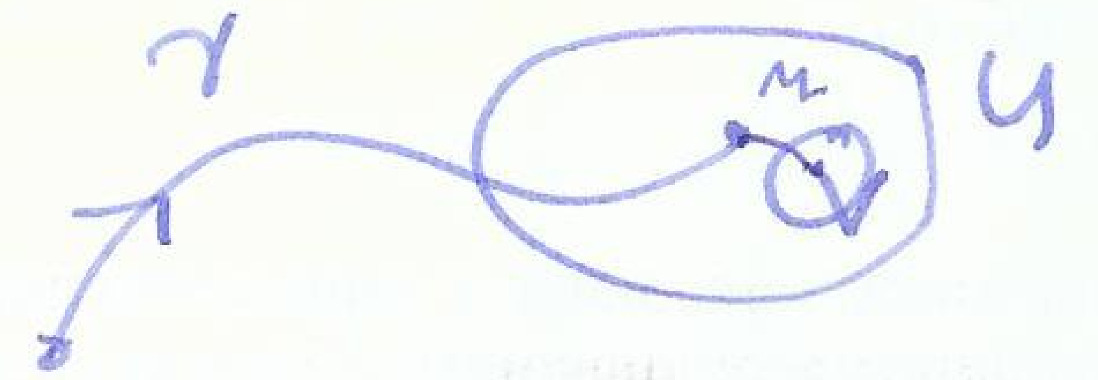
then $W_{[\gamma'']} \subset U_{[\gamma'']} \cap V_{[\gamma'']}$.



$p: U_{[\gamma]} \to U$ is a homeomorphism

$V_{[\gamma']} \longleftrightarrow V$ open for $[\gamma'] = [\gamma\eta]$

η path in U.



$p: \tilde{X} \to X$ continuous.

covering space: $p^{-1}(U) = \bigsqcup_{[\gamma]\in\pi_1 X} U_{[\gamma]}$ as if $U_{[\gamma]} \cap U_{[\gamma']}$ then $U_{[\gamma']} = U_{[\gamma]}$.

Finally: $\tilde{X}$ simply connected:

Let $[\gamma] \in \tilde{X}$ ~~the~~, and let γ_t be path along γ from $\gamma(0)$ to $\gamma(t)$

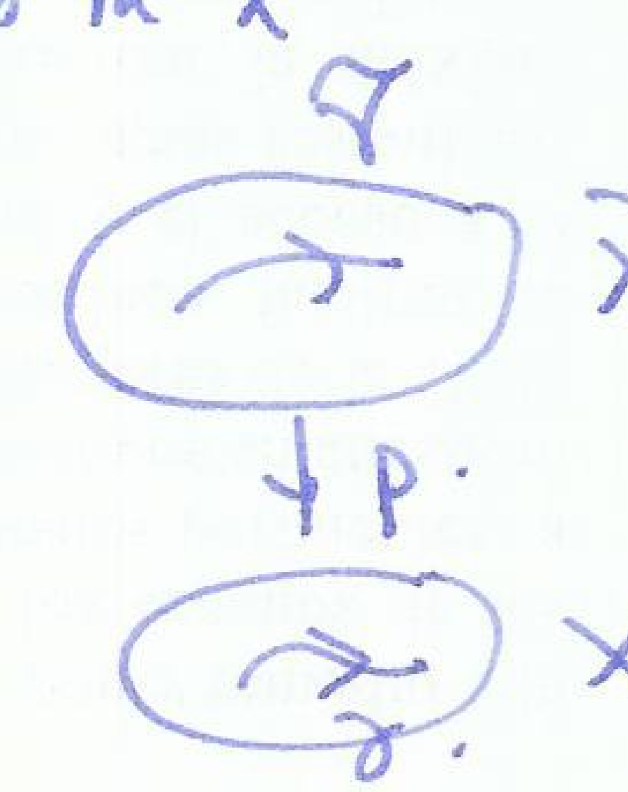
then γ_t is a path from $[\overset{x_0}{\text{const}}]$ to $[\gamma]$ so $\tilde{X}$ path connected.

consider $p_*(\pi_1(\tilde{X}, \tilde{x}))$ = loops in X that lift to loops in $\tilde{X}$

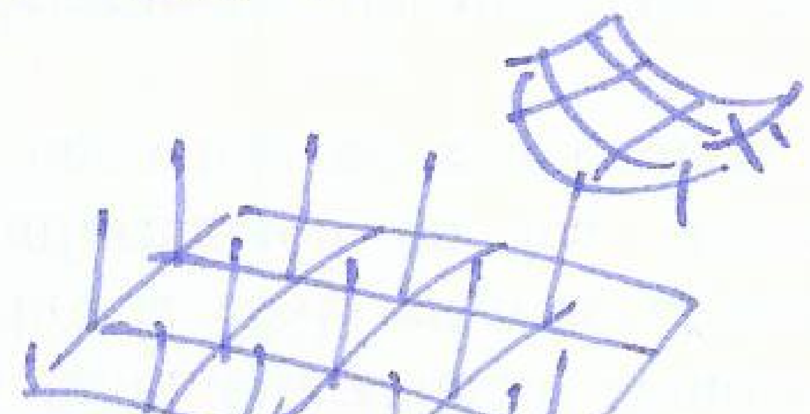
a loop in $\tilde{X}$ starts at $[x_0]$ and ends at $[x_0]$

endpoint of path $\tilde{\gamma}$ is $[\gamma_1]$ so $[\gamma_1] = [x_0]$

$\Rightarrow \gamma$ null homotopic, as required. $\square$.



<u>Examples</u> $T^2 \vee S^1$



Klein bottle

<u>Prop^n</u> X path connected, locally path connected, semilocally simply connected.

Then for every subgroup $H \subset \pi_1(X, x_0)$ there is a covering space

$p: \tilde{X}_H \to X$ s.t. $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$.

<u>Proof</u> Construct a quotient of $\tilde{X}$: let $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$

and $[\gamma\bar{\gamma}'] \in H$.

$\sim$ is an equivalence relation: reflexive ✓

symmetric ✓

transitive: H closed under multiplication

$[\gamma] \sim [\gamma']$ $\quad [\gamma\bar{\gamma}'] \in H$

$[\gamma'] \sim [\gamma'']$. $\quad [\gamma'\bar{\gamma}''] \in H$ $\quad [\gamma\bar{\gamma}'\gamma'\bar{\gamma}''] = [\gamma\bar{\gamma}''] \in H$.