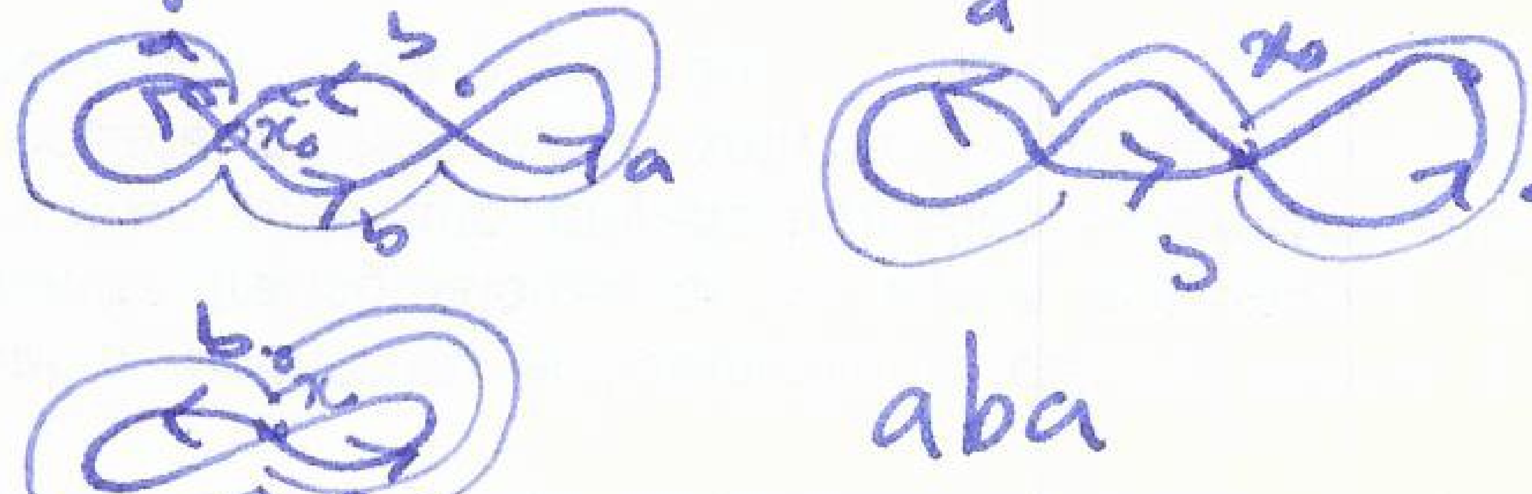
Corollary (Path lifting property).

Any path $f: I \to X$ starting at x_0 has a unique lift $\tilde f: I \to \tilde X$ starting at $\tilde x_0 \in p^{-1}(x_0)$. $\square$

Remark the lift of a constant loop is a constant loop.

Example. $\tilde f$.

$I \xrightarrow{f}$



aba

Prop$^{\underline{n}}$ Let $p: \tilde X \to X$ be a covering space. The map $p_*: \pi_1(\tilde X, \tilde x_0) \to \pi_1(X, x_0)$ is injective. The image subgroup $p_*(\pi_1(\tilde X, \tilde x_0)) < \pi_1(X, x_0)$ consists of homotopy classes of loops based at x_0, which lift to loops based at $\tilde x_0$.

Proof Let $\tilde f: I \to \tilde X$ be an element of the kernel of p_*

i.e. $p\tilde f$ is homotopic to the trivial loop $f_1: I \to x_0$ in X, by f_t say

but there is a lift $\tilde f_t: I \to \tilde X$, starting at $\tilde f$, and ending at a constant map. $\Rightarrow [\tilde f] = 0$ in $\pi_1(\tilde X, \tilde x_0) \Rightarrow p_*$ injective.

If $f: I \to X$ ~~has~~ is a loop with lift $\tilde f: I \to \tilde X$ which is a loop at $\tilde x_0$.

then $[f] \in p_*(\pi_1(\tilde X, \tilde x_0))$. Conversely, suppose $f': I \to X$ is a loop homotopic, by f_t, to a loop f^3 having such a lift $\tilde f$, then the homotopy lifts to $\tilde f_t$ so $\tilde f_1 = \tilde f'$ is a lift of f_1 which is a loop. $\square$.

Remark Let $p: \tilde X \to X$ be a covering space. Then $|p^{-1}(x)|$ is locally constant over $x \in X$. If $\tilde X$ connected then $|p^{-1}(x)|$ constant. This is called the *degree* or *number of sheets* of the cover.

<u>Prop^n</u>: Let $p : \tilde{X} \to X$ be a covering space with $X, \tilde{X}$ path connected.

then $\deg(p)$ = index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$. $[p_*(\pi_1(\tilde{X}, \tilde{x}_0)) : \pi_1(X, x_0)]$

<u>Proof</u> $\tilde{X}$ $\quad \tilde{x}_0 \quad \tilde{g} \quad p^{-1}(x_0)$.

$\downarrow p$

$X \quad x_0 \quad g$

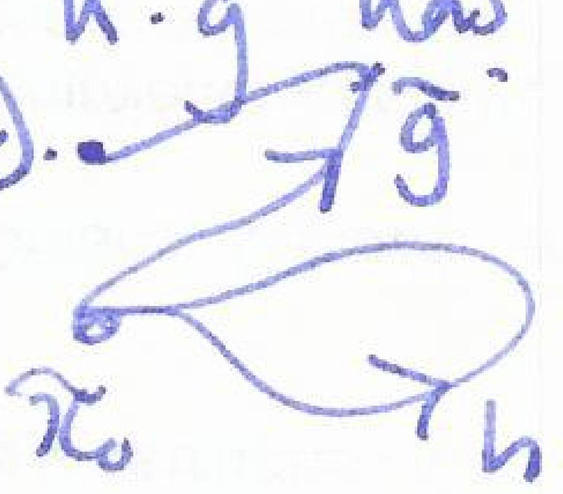
$g : I \to X$ loop based at x_0 has a lift

$\tilde{g} : I \to \tilde{X}$ <u>path</u> based at $\tilde{x}_0$ ending at $\tilde{g}(1) \in p^{-1}(x_0)$.

set $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ (<u>note</u>: if $[h] \in H$ then $\tilde{h}$ is a <u>loop</u> so $\tilde{h} : I \to \tilde{X}$ ends at $\tilde{x}_0$).

so for any $[h] \in H$ $\quad h \cdot g$ has a lift $\tilde{h} \cdot \tilde{g}$ ending at $\tilde{g}(1)$.

as $\tilde{h}$ is a <u>loop</u> based at $\tilde{x}_0$.



so we can define a function Φ : cosets of $H \longrightarrow p^{-1}(x_0)$.

$$H[g] \longmapsto \tilde{g}(1).$$

<u>claim</u> Φ : surjective : $\tilde{X}$ path connected, so there is a path connecting $\tilde{x}_0$ to any other point in $p^{-1}(x_0)$.

Φ injective : suppose $\Phi(H[g_1]) = \Phi(H[g_2])$.

then $\tilde{g}_1(1) = \tilde{g}_2(1) \in p^{-1}(x_0)$ so $g_1 \bar{g}_2$ is a loop in $\tilde{X}$ based at x_0, so $[g_1][g_2]^{-1} \in H \Rightarrow H[g_1] = H[g_2]$. $\square$.

Q: when do maps lift? A lifting criterion/property

Prop^n Let $p:(\tilde{X},\tilde{x}_0) \to (X,x_0)$ be a covering space, and let $f:(Y,y_0) \to (X,x_0)$, Y path connected, locally path connected.

Then a lift $\tilde{f}:(Y,y_0) \to (\tilde{X},\tilde{x}_0)$ exists iff $f_*(\pi_1(Y,y_0)) \subset p_*(\pi_1(\tilde{X},\tilde{x}_0))$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd} \qquad f_*(\pi_1 Y) \subset p_*(\pi_1 \tilde{X}).$$

Def^n X is locally path connected if for each $x \in X$, and each open nbd $x \in U \subset X$, there is an open nbd $x \in V \subset U \subset X$ w/ V path connected.

Proof $\Rightarrow$ if the lift exists then $p\tilde{f} = f$ so $p_*\tilde{f}_* = f_*$

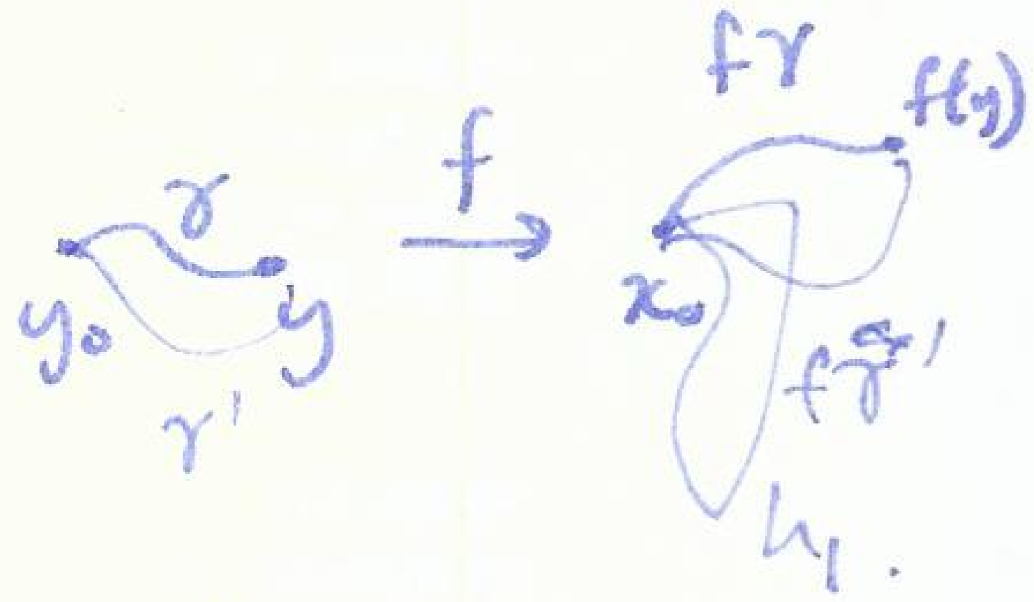
so $f_*(\pi_1 Y) = p_*\tilde{f}_*(\pi_1 Y) \subset p_*(\pi_1 \tilde{X})$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$\Leftarrow$ assume: $f_*(\pi_1(Y,y_0)) \subset p_*(\pi_1(\tilde{X},\tilde{x}_0))$

let $y \in Y$, and let γ be a path in Y from y_0 to y.

there is a unique lift $\widetilde{f\gamma}$ starting at $\tilde{x}_0$



define $\tilde{f}(y) = \widetilde{f\gamma}(1)$

check: well defined. spose γ' is some other path from y_0 to y. Then $h_0 = f\gamma' \cdot \overline{f\gamma}$ is a loop at x_0 with

$[h_0] \in f_*(\pi_1(Y,y_0)) \subset p_*(\pi_1(\tilde{X},\tilde{x}_0))$

so there is a homotopy h_t of h_0 to a loop h_1, that lifts to a loop $\tilde{h}_1$ in $\tilde{X}$ based at $\tilde{x}_0$

apply <u>covering homotopy property</u> to lift this homotopy to $\tilde{h}_t$