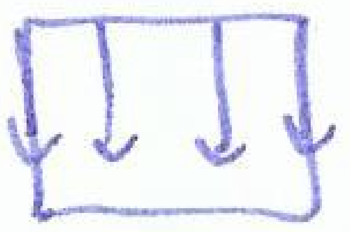
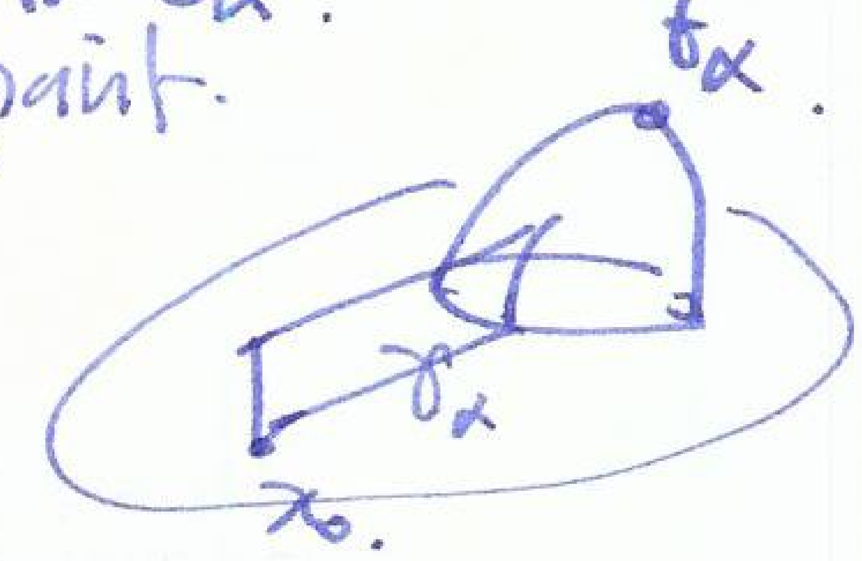
top edges not attached to anything so deformation retraction is:

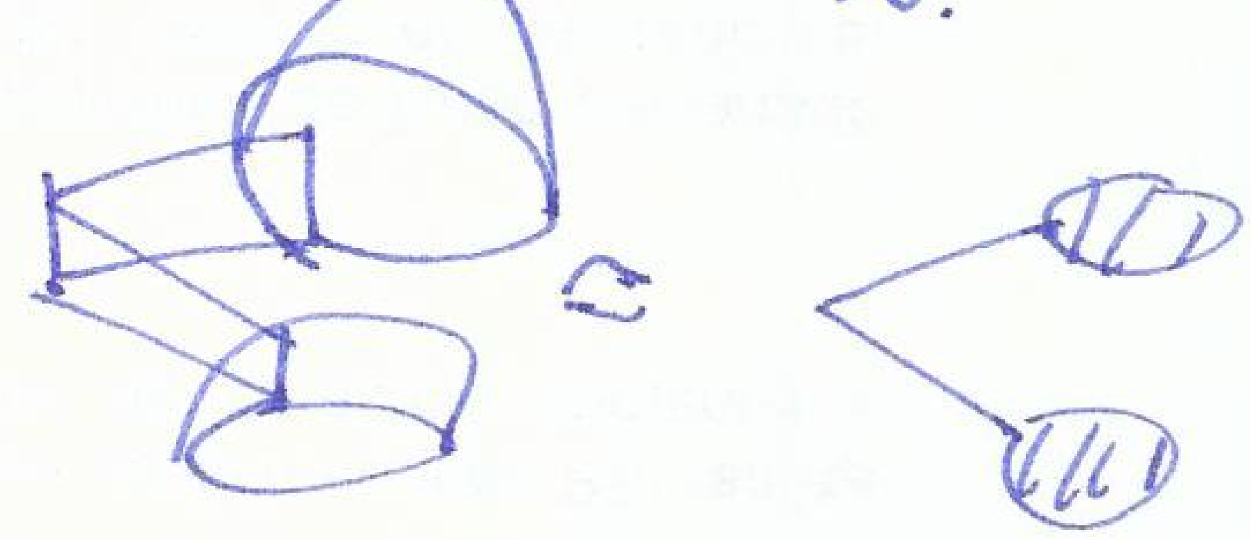


In each e^2_α choose a point $t_\alpha \neq t_0$ basepoint. & attaching arc for S_α.



set $A = Z \setminus \bigcup_\alpha t_\alpha$ ← deformation retracts onto X.

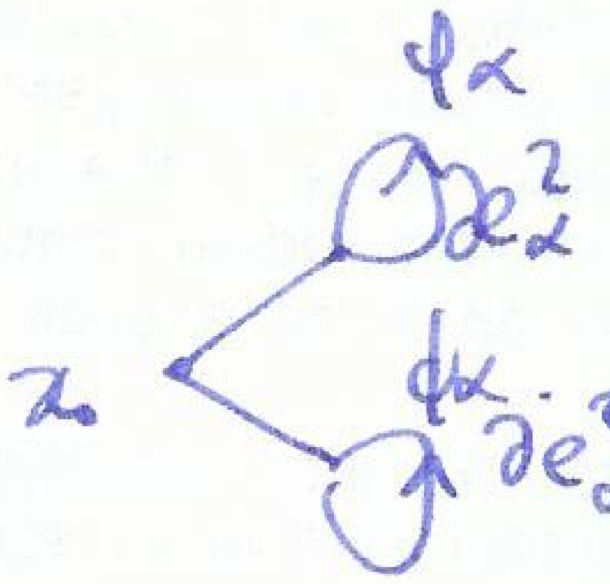
$B = Z \setminus X$. ← contractible!



van Kampen: $\pi_1 Y \cong \pi_1(A) * \pi_1(B) / N \cong \pi_1(A)/N$.

where N = normal subgroup generated by image of $\pi_1(A \cap B) \xrightarrow{i_*} \pi_1(A)$.

$$\begin{tikzcd} & A \arrow[dr, hook] & \\ A\cap B \arrow[ur, hook] \arrow[dr, hook] & & Z \\ & B \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(A) \arrow[dr] & \\ \pi_1(A\cap B) \arrow[ur] \arrow[dr] & & \pi_1(Z) \\ & \pi_1(B) \arrow[ur] & \end{tikzcd}$$

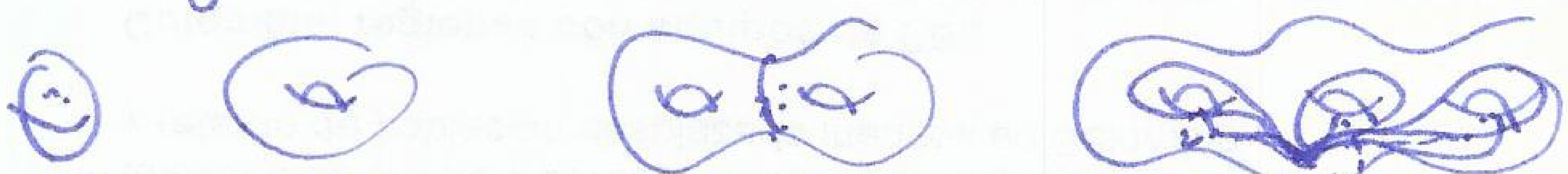


$\pi_1(A \cap B)$ deformation retracts to a graph

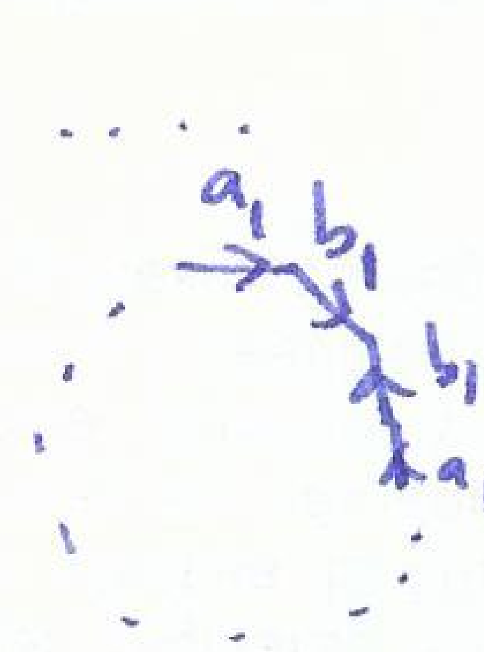
where $\bigcup \gamma_\alpha$ = maximal tree and each ∂e^2_α is an edge not in the maximal tree so $\pi_1(A \cap B)$ is free on $\langle \gamma_\alpha \phi_\alpha \bar{\gamma}_\alpha \rangle$. D.

~~Corollary~~ Fact (Classification of closed surfaces).

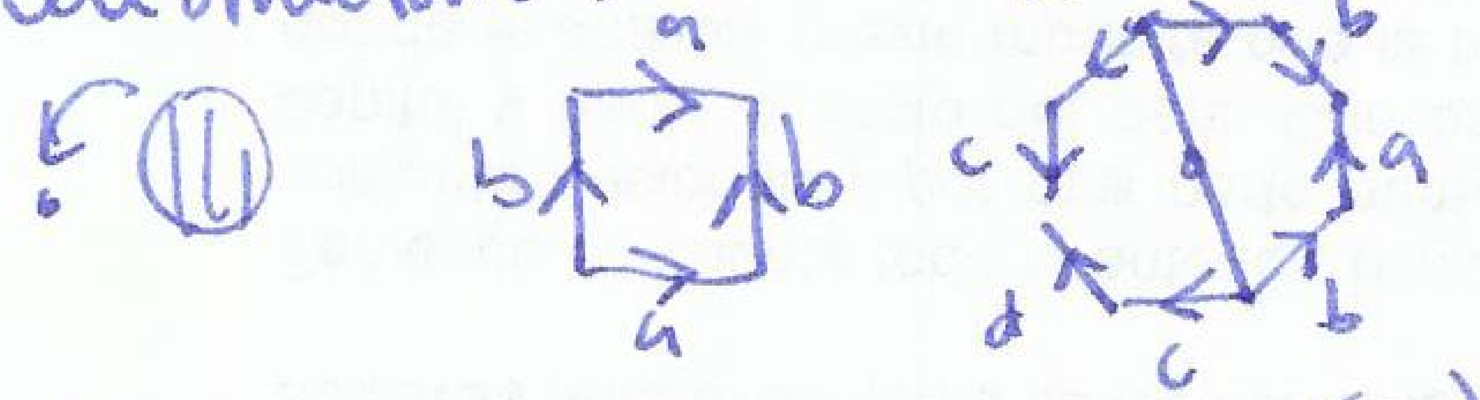
Fact: every closed orientable surface is of the form



S^2 $\quad T^2$ $\quad S_2$ $\quad S_3$



with cell structure:



etc.

$\pi_1(S^2) = 1$. $\quad \pi_1(T^2) = \langle a, b \mid [a,b] \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$

$\pi_1(S_2) = \langle a, b, c, d \mid [a,b][c,d] \rangle$.

$\pi_1(S_g) = \langle a_i, b_i \mid [a_1,b_1][a_2,b_2] \cdots [a_g,b_g] \rangle$.

Corollary These surfaces are all genuinely different!

$$ab(\pi_1(S_g)) \cong \mathbb{Z}^{2g}.$$

Corollary For every group G there is a 2-dimensional cell complex X_G with $\pi_1(X_G) = G$.

Proof choose a presentation $G = \langle g_\alpha \mid r_\beta \rangle$

$X^{(0)} = \{x_0\}$ $\quad X^{(1)} = \{x_0\} \cup e^1_\alpha$ ← one edge for each generator

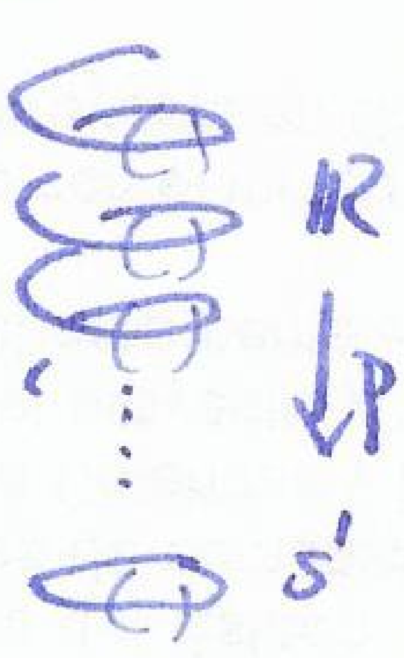
$X^{(2)} = X^{(1)} \cup_{\phi_\alpha} e^2_\beta$. $\quad (\simeq \bigvee_\alpha S^1_\alpha)$.

← one 2-cell for each relator, where $\phi_\beta : S^1 = \partial e^2_\alpha$ to path r_β $\square$

Corollary Can't classify : 2-complexes ; 4-manifolds.

§1.3 Covering spaces

Example

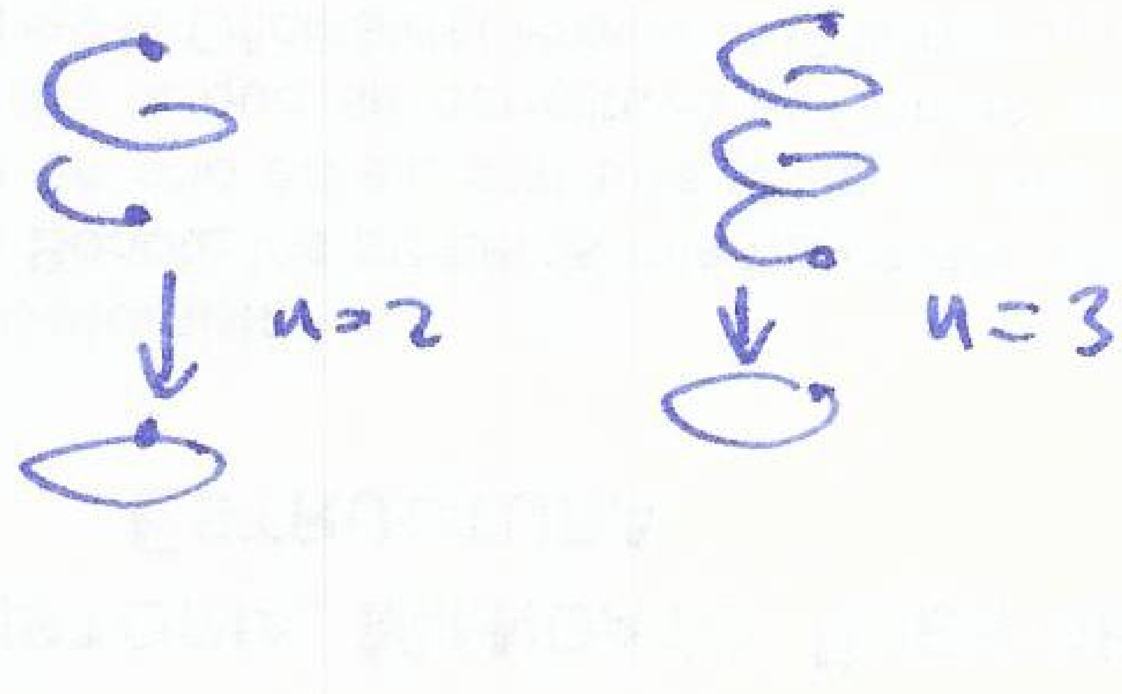


property (*): there is an open cover U_α of S^1 s.t. $p^{-1}(U_\alpha)$ is homeomorphic to a discrete set $D \times U_\alpha$.

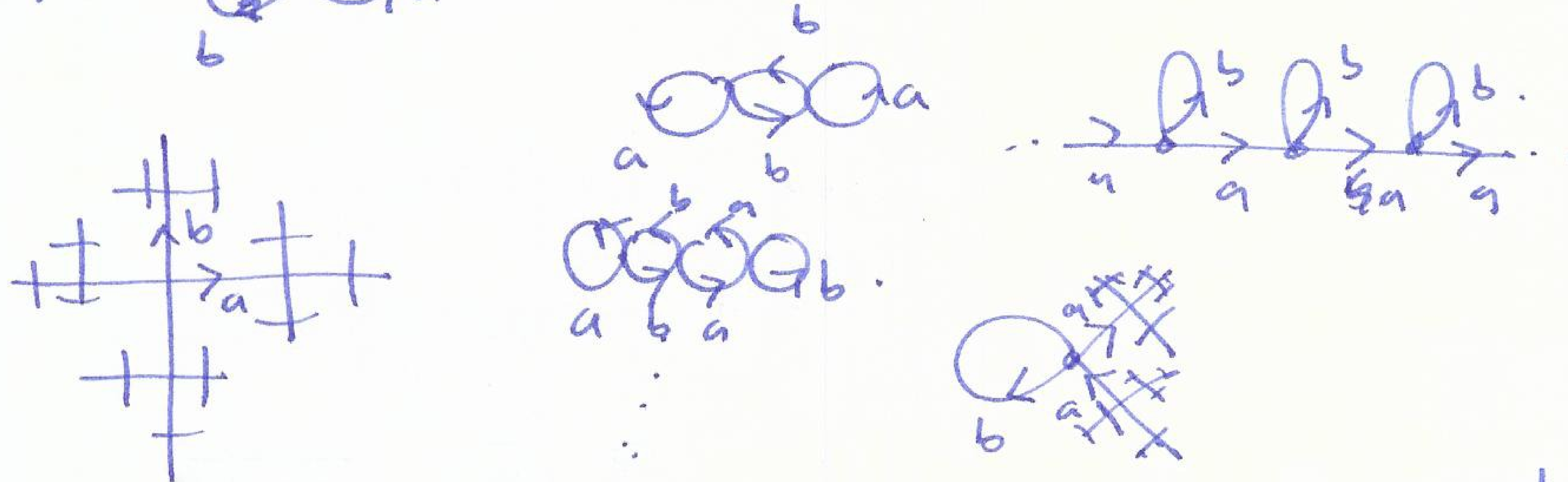
Def^n A covering space of X is a space $\tilde{X}$, and a map $p : \tilde{X} \to X$ such there is an open cover U_α of X s.t. $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each mapped homeomorphically to U_α by p.

Example $p : S^1 \to S^1$

$z \mapsto z^n$



Examples 



basically, any 4-valent graph with oriented labelled edges: 

Remark a covering space $\tilde{X} \xrightarrow{p} X$ gives rise to a subgroup of $\pi_1 X$.

$$p_*(\pi_1(\tilde{X})) < \pi_1(X)$$

Fact: the map p_* is injective (proof later).

and there is a 1-1 correspondence between based covering spaces $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and subgroups of $\pi_1(X, x_0)$.

Lifting properties

recall:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

a lift of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ s.t. $p\tilde{f} = f$.

Prop^n (Homotopy lifting property)

Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$ and a map $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0, there is a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ lifting f_t.

Proof we proved this for $p : \mathbb{R} \to S^1$ only using property $(*)$ $\square$.