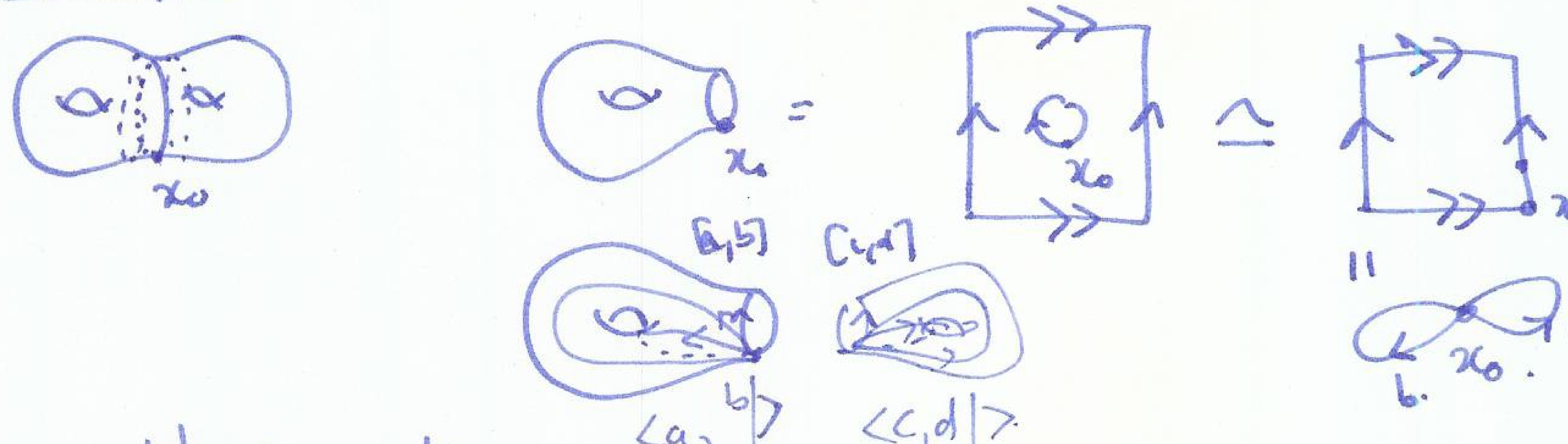
now homotopy lower path to upper path across the square in $A_{\alpha_{k+1}}$, this is a type ① operation. continue across all squares $\square$.

① $\pi_1(S^1 \vee S^1) = F_2$. 

Applications/Examples.

② $\pi_1(S_2)$



$$\pi_1(S_2) \cong \langle a,b|\rangle * \langle c,d|\rangle / N \qquad N = \langle [a,b]=[c,d] \rangle.$$

$$\cong \langle a,b,c,d \mid [a,b][c,d] \rangle$$ (standard presentation for $\pi_1(S_2)$).

Cell complexes

n-cell: closed n-ball 

① start with a discrete set X^0 the 0-cells (points).

② form the X^{n+1} skeleton by attaching cells to X^n by gluing them along their boundaries.

formally: choose $(n+1)$-cells D_α^{n+1} and gluing/attaching maps $\phi_\alpha : \partial D_\alpha^{n+1} \longrightarrow X^n$, then form quotient space $X^n \coprod_\alpha D_\alpha^{n+1} / \sim$
$\partial D_\alpha^{n+1} \cong S^n$

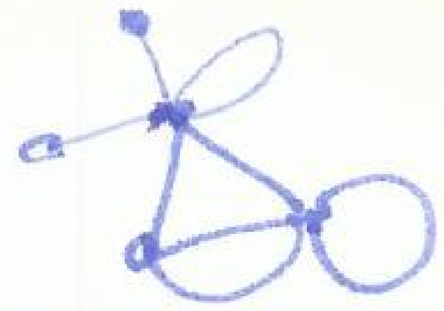
where $x \sim \phi_\alpha(x)$ for all $x \in \partial D_\alpha^{n+1}$

③ stop at stage X^n: finite dimensional cell complex (cw complex)

or consider $X = \bigcup_n X^n$ ← give this the weak topology

$A \subset X$ open if $A \cap X^n$ open for each n.

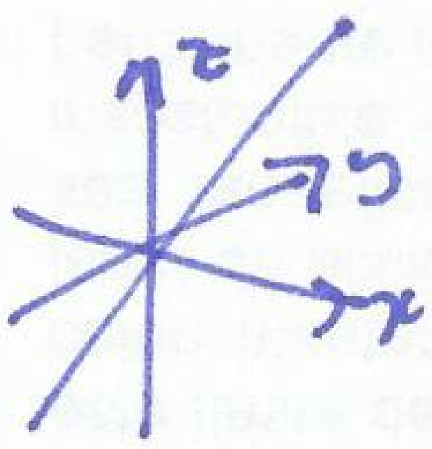
Examples

graphs X^0 vertices, X^1 edges 

1-cell is an edge $e_1 \cong I$, $\partial e_1 = \{0,1\}$ map to X^0.

S^n: $X^0 = \{pt\}$. X^n $\underset{\substack{\uparrow \\ \text{0-cell}}}{e_0} \cup \underset{\substack{\uparrow \\ \text{n-cell}}}{e_n}$ attaching map $\partial e_n \xrightarrow{\phi} pt$
e_0 (equivalent to $S^n = D^n/\partial D^n$).

S^1:  S^2: 

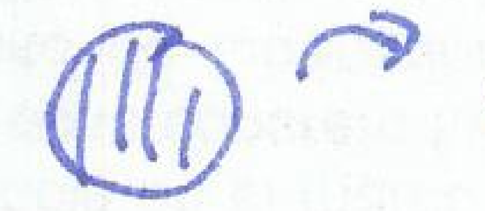
$\mathbb{R}P^n$  space of lines through the origin in $\mathbb{R}^{n+1}$

topology: $\mathbb{R}^{n+1}/\{0\}/\sim$ $\quad v \sim \lambda v$ $\quad \lambda \in \mathbb{R}$. $\lambda \neq 0$

equivalently: $\mathbb{R}P^n = S^n/\sim$ $\quad x \sim -x$

i.e. $\mathbb{R}P^n = D^n/\sim$ identifies antipodal points on ∂D^n

note: $\partial D^n/\sim$ antipodal map $= \mathbb{R}P^{n-1}$, so $\mathbb{R}P^n$ made from attaching a n-cell to $\mathbb{R}P^{n-1}$ by the quotient map $S^{n-1} \to \mathbb{R}P^{n-1}$ (2-1 map), so $\mathbb{R}P^n$ has a cell structure with 1 cell in each dimension.

$\mathbb{R}P^0$ $\cdot$ $\quad \mathbb{R}P^1$  $= S^1$ $\quad \mathbb{R}P^2$  $\mathbb{R}P^1$, etc.

van Kampen: $\pi_1(\text{graph}) =$ free group.

choose maximal tree T  $\Gamma = T \cup e_i$ edges not in maximal tree.

Let $A_i = N_\epsilon(T) \cup e_i$

$\bigcap A_i = N(T)$ contractible

$A_i \simeq S^1$ so $\pi_1(A_i) \cong \mathbb{Z}$

so $\pi_1(\Gamma) \cong \ast_\alpha \pi_1(A_i)/N$ N trivial.

$\pi_1(A_i) = \mathbb{Z}$

$\pi_1(\Gamma) \cong F_n$ free group on n-generators $\leftrightarrow$ n-edges not in maximal tree.

$\underline{\pi_1(S^n)}$ $(n \geq 2)$

cell structure with two cells ~~e_0, e_n~~ in each dimension

so $S^n = N(e^n_1) \cup N(e^n_2)$. with $N(e_n) \cap N(e_n') = N(S^{n-1})$.

$N(S^{n-1})$ path connected as long as $n \geq 2$.

so $\pi_1(S^n) \cong \pi_1(N(e_n)) * \pi_1(N(e_n'))/N \cong$ trivial group

~~$\pi_1(\mathbb{R}P^n)$~~ **Fact** ① X cell complex, $\pi_1(X)$ depends only on the 2-skeleton $X^{(2)}$, higher dimensional cells do not change π_1.

② we can write down a presentation for $\pi_1(X)$ given $X^{(2)}$.

Prop^n- X path connected $Y = X \cup_{\phi_\alpha} e^2_\alpha$ $\phi_\alpha : \partial e^2_\alpha \to X$ gluing map.

choose path γ_α from basepoint $x_0 \in X$ to $\phi_\alpha(s_0)$ $s_0 \in S^1$ basepoint.

Let $N =$ normal subgroup generated by $\gamma_\alpha \phi_\alpha \bar{\gamma}_\alpha$
$= \langle \gamma_\alpha \phi_\alpha \bar{\gamma}_\alpha \rangle$.

Then $\pi_1(Y, x_0) \cong \pi_1(X, x_0)/N$.

Proof (van Kampen!) Replace Y by a space Z with each path "thickened up" to $\gamma_\alpha \times I$ (Z deformation retracts to Y).

i.e. attach rectangle $R_\alpha = I \times I$
with $I \times \{0\}$ attached to γ_α.
$\{0\} \times I$ all identified
$\{1\} \times I$ attached to an arc in e^2_α.