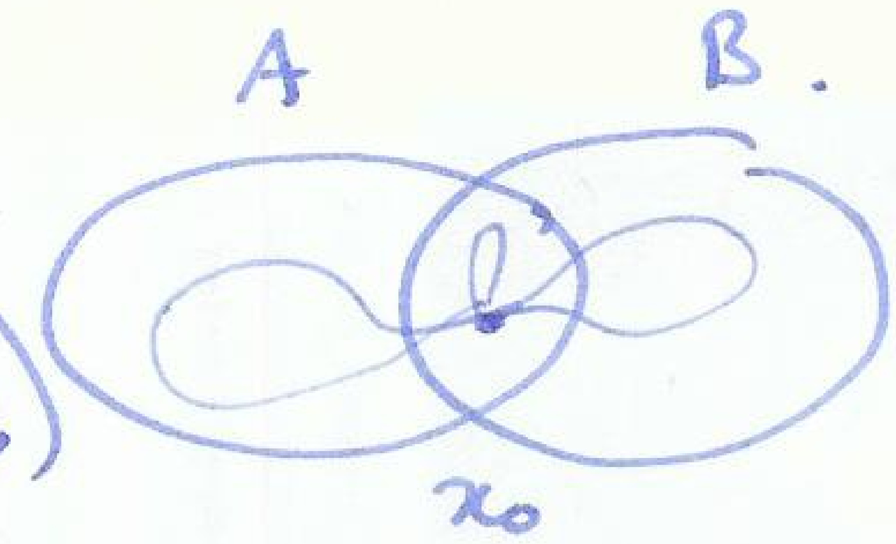
van Kampen : idea $X = A \cup B$

$A \hookrightarrow A \cup B \qquad B \hookrightarrow A \cup B$



A, B path connected.

$A \cap B$ path connected.

$\pi_1(A, x_0) \to \pi_1(X, x_0) \qquad \pi_1(B, x_0) \to \pi_1(X, x_0)$

$$\pi_1(A, x_0) * \pi_1(B, x_0) \overset{\Phi}{\twoheadrightarrow} \pi_1(X, x_0).$$
surjective.

$\ker(\Phi) =$ normal subgp generated by $i_\alpha(\omega) i_\beta(\omega)^{-1}$.

$$\begin{tikzcd} & A \arrow[dr, "i_\alpha"] & \\ A \cap B \arrow[ur] \arrow[dr] \arrow[rr, bend left] \arrow[rr, bend right] & & A \cup B = X \\ & B \arrow[ur, "i_\beta"'] & \end{tikzcd}$$

Th^{m} (two set version) If $X = A \cup B$ with basepoint $x_0 \in A \cap B$, A, B open, path connected, and $A \cap B$ path connected, then $\Phi : \pi_1(A, x_0) * \pi_1(B, x_0) \to \pi_1(X, x_0)$ is surjective, with kernel normal subgroup N generated by all elements of the form $i_\alpha(\omega) i_\beta(\omega)^{-1}$, i.e. so ~~$\Phi$: π_1 ... $\pi_1(X, x_0)$~~

$$\pi_1(X, x_0) \cong \pi_1(A, x_0) * \pi_1(B, x_0) / N.$$

Th^{m} (general version) Let X be the union of open path connected sets A_α, each containing the basepoint x_0, and $A_\alpha \cap A_\beta$ path connected, then $\Phi : \mathop{*}_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ is surjective. Furthermore, if $A_\alpha \cap A_\beta \cap A_\gamma$ path connected, then $\ker \Phi = N$ normal subgroup generated by:

$$\begin{tikzcd} & A_\alpha \arrow[dr, "i_{\alpha\beta}"] & \\ A_\alpha \cap A_\beta \arrow[ur, hook] \arrow[dr, hook] \arrow[rr, bend left] \arrow[rr, bend right] & & X \\ & A_\beta \arrow[ur, "i_{\beta\alpha}"'] & \end{tikzcd} \qquad i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}.$$

and so $\pi_1(X, x_0) \cong \mathop{*}_\alpha \pi_1(A_\alpha, x_0) / N$.

Example Wedge sum (one point union) general $\bigvee_\alpha X_\alpha$.

$X \vee Y = X \cup Y / \sim$ with $x_0 \sim y_0$ and no other identifications.

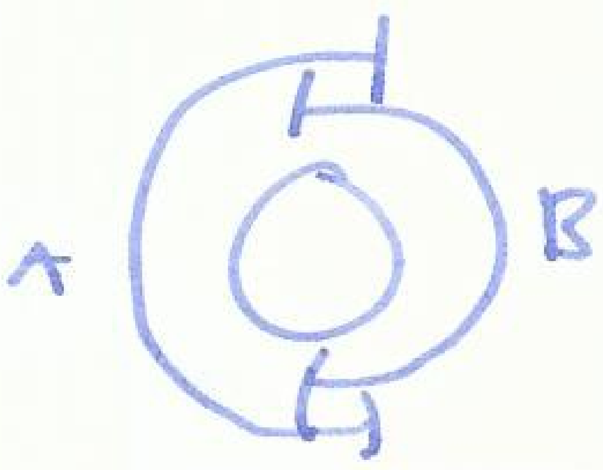
$S^1 \vee S^1$ "nice" wedge sums: $x_0 \in U_\alpha \subset X_\alpha$, U_α open nbd

$\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) / N$ $\qquad U_\alpha$ deformation retracts to x_0.

but $\{x_0\}$ has contractible nbd so N trivial, so $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$.

so $\pi_1(\underbrace{S^1 \vee \dots \vee S^1}_{k \text{ circles}}) \cong F_k$.

Non-example S^1 = union of two intervals



$A \cap B = \leftrightarrow \cup \leftrightarrow$ not path connected.

Example S^2 = union of two discs.

$A \cap B =$

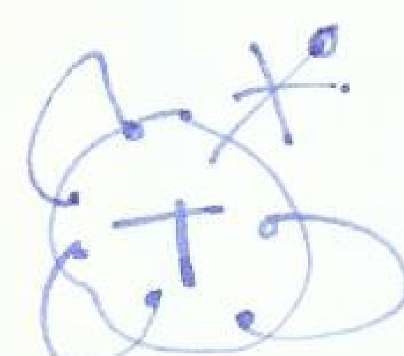
$S^1 \times I$ path connected.

so $\pi_1(S^2) \cong \underbrace{\pi_1(A) * \pi_1(B)}_{\text{trivial}} / N \cong 1$.

Example X connected graph, then $\pi_1(X)$ free.

$X =$

choose a maximal tree $T \subset X$



exercise: T contains all vertices of X.

set A_α = open nbd of $T \cup$ a single edge.

claim: we can apply van-Kampen: $A_\alpha \cap A_\beta$ = open nbd of T. (path connected).

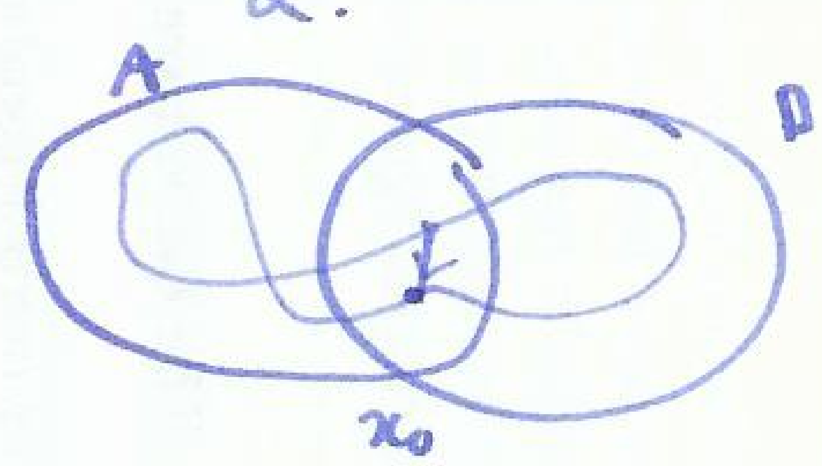
$A_\alpha \cap A_\beta \cap A_\gamma$ =

$$\pi_1(X) \cong \mathop{*}_{\alpha} \pi_1(A_\alpha) / N.$$

$\pi_1(A_\alpha) \neq A_\alpha \simeq T \cup$ single edge $\simeq S^1$ so $\pi_1(A_\alpha) \cong \mathbb{Z}$.

$\pi_1(A_\alpha \cap A_\beta)$ trivial $\Rightarrow \pi_1(X) \cong \mathop{*}_{\alpha} \mathbb{Z}$. free product of $\mathbb{Z}$'s.

Proof (of van Kampen) recall idea:



setup $X = \bigcup A_\alpha$ A_α open, $x_0 \in A_\alpha$

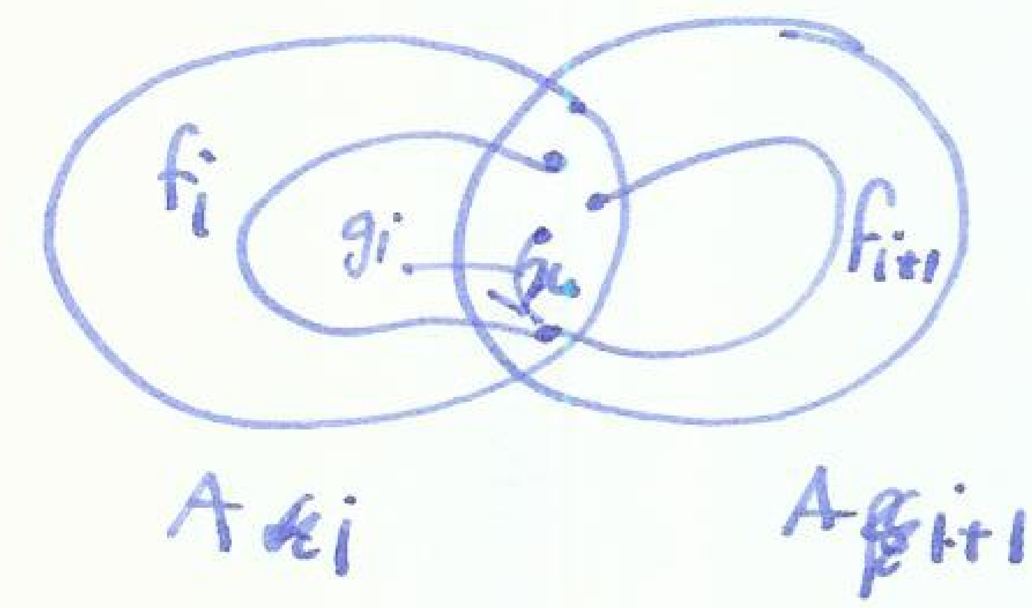
$A_\alpha \cap A_\beta$, $A_\alpha \cap A_\beta \cap A_\gamma$ path connected. $\Phi : * \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$.

Claim Φ surjective:

Proof let $f : I \to X$ be a loop. claim: there is a partition $0 = s_1 < s_2 < \dots < s_n = 1$ of $[0,1]$ s.t. for each $[s_i, s_{i+1}]$, $f([s_i, s_{i+1}]) \subset A_\alpha$ for some single A_α.

reason: U_α open cover of X so $f^{-1}(U_\alpha)$ open cover of I, compact, so there is a finite subcover. □.

let f_i be $f|_{[s_i,s_{i+1}]}$ and let A_i be the A_α s.t. $f([s_i,s_{i+1}]) \subset A_\alpha$



so $f = f_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_n$

↑ path composition.

$A_i \cap A_{i+1}$ path connected, so there is a path g_i in $A_i \cap A_{i+1}$ from x_0 to $f_i(s_i) = f_{i+1}(s_i)$ ~~$f_i(s_i) = f_{i+1}(s_i)$ to x_0~~.

$$f \simeq \underbrace{f_1 \cdot \bar{g}_1}_{\subset A_1} \cdot \underbrace{g_1 \cdot f_2 \cdot \bar{g}_2}_{\subset A_2} \cdot \underbrace{g_2 \cdot f_3 \cdot \bar{g}_3}_{\subset A_3} \cdots \underbrace{g_{n-1} g_n}_{\subset A_n} = h_1 h_2 \dots h_k \in \mathop{*}_\alpha \pi_1(A_\alpha).$$

for $h_i \in \pi_1(A_i)$.

so $[f]$ lies in the image of Φ as required □.

we have shown $\Phi : \mathop{*}_\alpha \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$ surjective.

identify $\ker(\Phi) = N = \langle\, i_{\alpha\beta}(f)\, i_{\beta\alpha}(f)^{-1} \,\rangle$ set $Q = \mathop{*}_\alpha \pi_1(A_\alpha, x_0)/N$

<u>Notation</u>: A <u>factorization</u> of $[f] \in \pi_1(X, x_0)$ is a formal product $[f_1][f_2] \cdots [f_k]$ where f_i is a loop in same (A_α, x_0) and $[f_i]$ is the homotopy class of f_i $f \simeq f_1 \cdot f_2 \cdots f_k$ in X, i.e. $[f_1] \cdots [f_n]$ is an (unreduced) word in $\pi_1(X, x_0)$ s.t. $\Phi([f_1] \cdots [f_k]) = [f]$. Surjectivity $\Rightarrow$ every $[f]$ has a factorization.

Two factorizations are <u>equivalent</u> if related by the following operations:

① if ~~adjont~~ adjacent terms $[f_i][f_{i+1}]$ lie in same A_α replace them with $[f_i \cdot f_{i+1}]$

② if f_i is a loop in $A_\alpha \cap A_\beta$ replace $[f_i] \in \pi_1(A_\alpha)$ with $[f_i] \in \pi_1(A_\beta)$.

<u>Remark</u> ①: does not change element of $\mathop{*}_\alpha \pi_1(A_\alpha, x_0)$

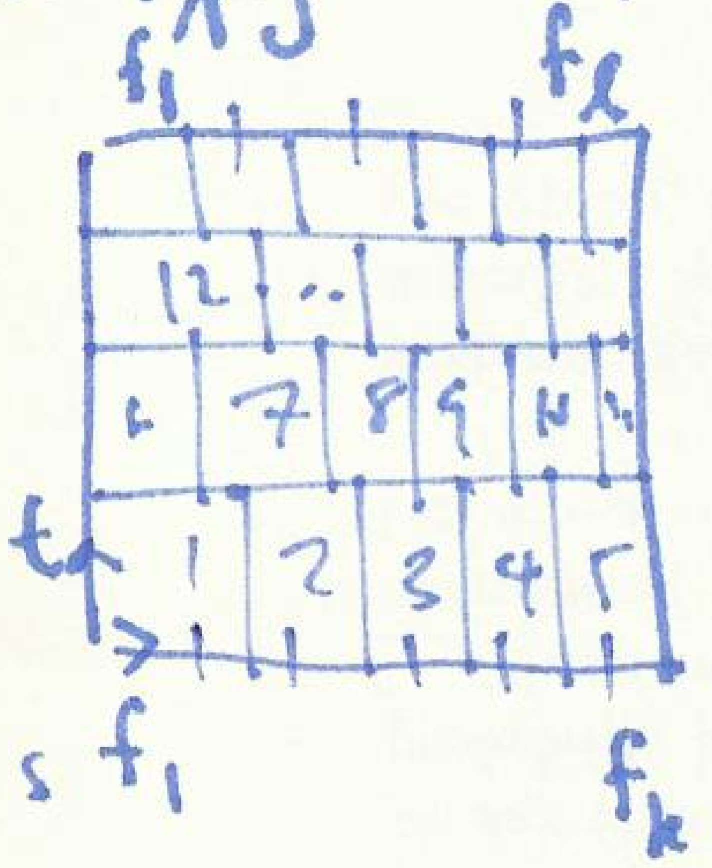
②: does not change image in ~~$\pi_1(X, x_0)$~~ $Q = \mathop{*}_\alpha \pi_1(A_\alpha, x_0)/N$.

i.e. equivalent factorizations give same image in Q.

aim: (suffices to show) any two factorizations of $[f]$ are equivalent by ①, ②

this implies $Q \xrightarrow{\Phi} \pi_1(X, x_0)$ is injective. so $Q \cong \pi_1(X, x_0)$.

let $[f_1] \cdots [f_k]$ and $[f_1'] \cdots [f_\ell']$ be two factorizations of $[f]$, so they are homotopic (in X!). Let $F: I \times I \to X$ be such a homotopy.



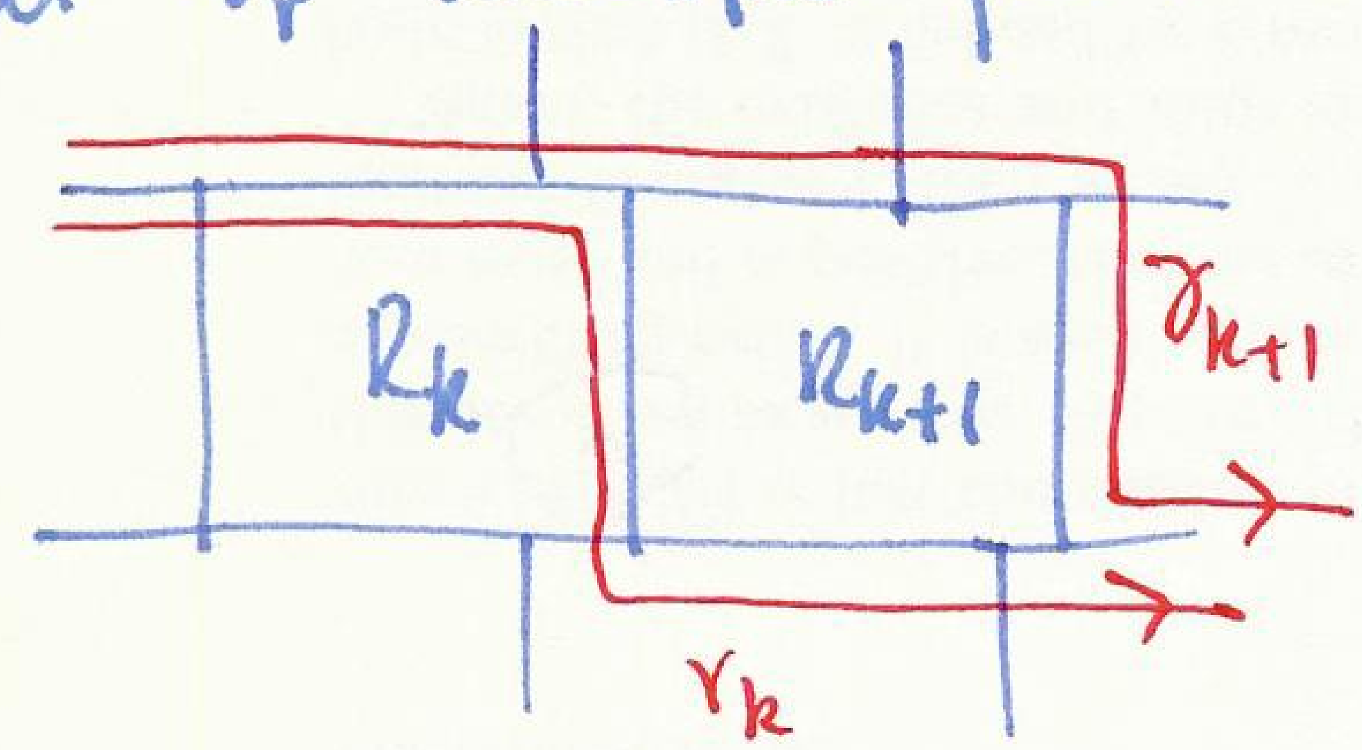
F is continuous, so we can partition F into finitely many rectangles, such that the image of each rectangle lies in a single A_α

wlog: can divide t direction into strips; then strips into rectangles such that at most 3 rectangles share a common vertex. — number $R_1 \dots R_N$

note: any path from left to right is a loop in (X, x_0)

let γ_r be the path separating $R_1 \dots R_r$ from $R_{r+1} \dots R_N$.

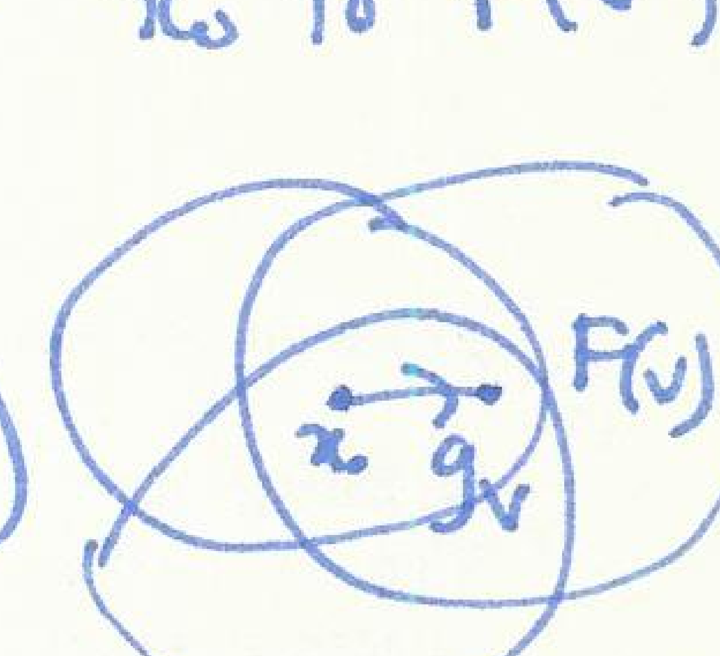
(all γ_r homotopic paths in (X, x_0)).

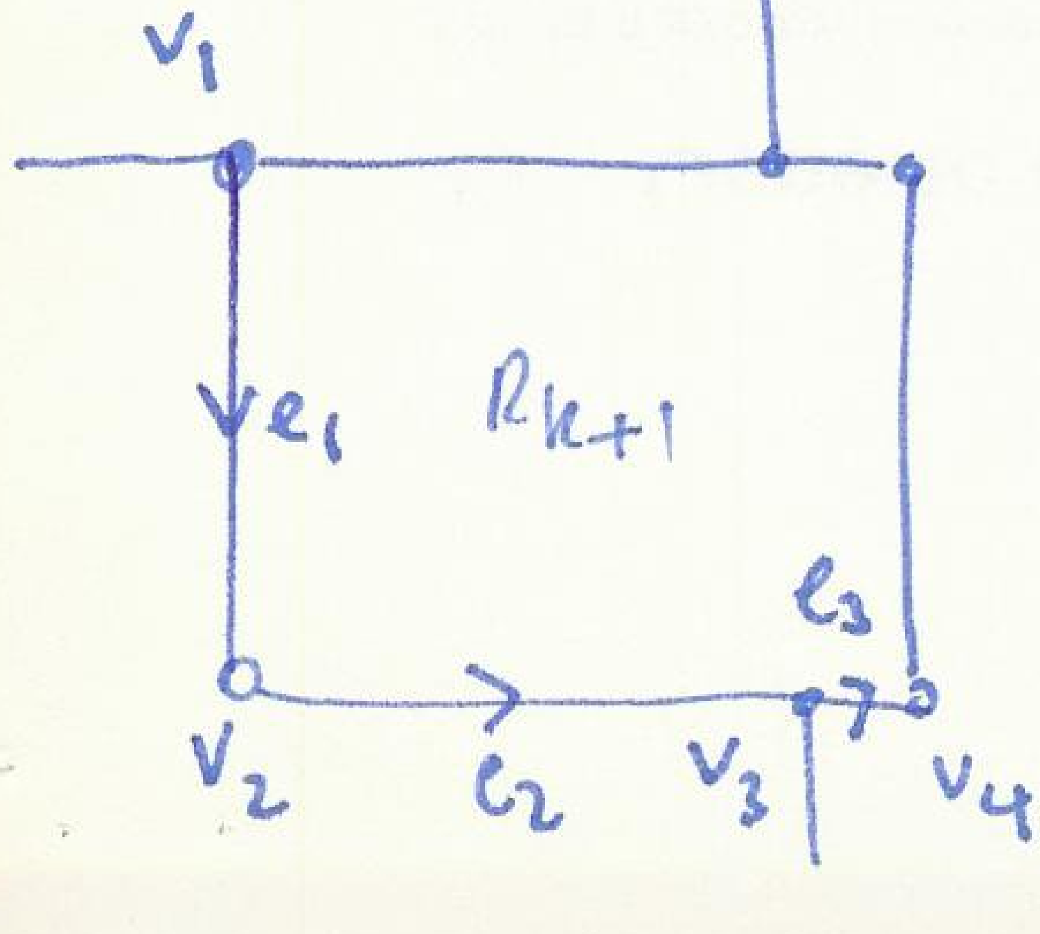


for each vertex v lies in 3 squares, so 3 sets $A_{\alpha_{v_1}}, A_{\alpha_{v_2}}, A_{\alpha_{v_3}}$

for each vertex v choose a path g_v from x_0 to $F(v)$ such that $g_v \subset A_{\alpha_{v_1}} \cap A_{\alpha_{v_2}} \cap A_{\alpha_{v_3}}$

(uses hypothesis: triple intersections path connected)





lower path γ_k:

$$\cdots \underbrace{\bar{g}_{v_1}, g_{v_1} e_1 \bar{g}_{v_2},}_{\substack{R_k \\ R_{k+1}}\ \supset ①} \underbrace{g_{v_2} e_2 \bar{g}_{v_3},}_{\substack{R_{k+1} \\ R_-}\ \supset ②} \underbrace{g_{v_3} e_3 \bar{g}_{v_4},}_{\substack{R_{k+1} \\ R_-}\ \supset ②} g_{v_4} \cdots$$