a homotopy of pairs / basepoint preserving homotopy is a family of maps

$\phi_t:(X,x_0)\to(Y,y_0)$ (i.e. $F:(X\times I, x_0\times I)\to(Y,y_0)$ cts.).

$(x,t)\longmapsto f_t(x)$

each f_t induces a map $(\phi_t)_*:\pi_1(X,x_0)\to\pi_1(Y,y_0)$

Prop$^{\underline{n}}$ $(\phi_0)_*=(\phi_1)_*$ Proof $f:I\to X$. $f:(I,\{0,1\})\to(X,x)$.

$(\phi_0)_*[f]=[\phi_0 f]=[\phi_t f]=[\phi_1 f]=(\phi_1)_*[f]$ □

Defn X,Y are homotopy equivalent if there are maps $X \underset{\psi}{\overset{\phi}{\rightleftarrows}} Y$
s.t. $\phi\psi\simeq id_Y$ and $\psi\phi\simeq id_X$.

$(X,x_0),(Y,y_0)$ are homotopy equivalent if there are maps

$(X,x)\underset{\psi}{\overset{\phi}{\rightleftarrows}}(Y,y_0)$ st. $\phi\psi\simeq id_{(Y,y_0)}$ $\psi\phi\simeq id_{(X,x)}$.

i.e. homotopies fix basepoint!

Prop$^{\underline{n}}$ If $\phi:X\to Y$ is a homotopy equivalence, then $\phi_*:\pi_1(X,x_0)\to\pi_1(Y,\phi(x_0))$ is an isomorphism for all $x_0\in X$.

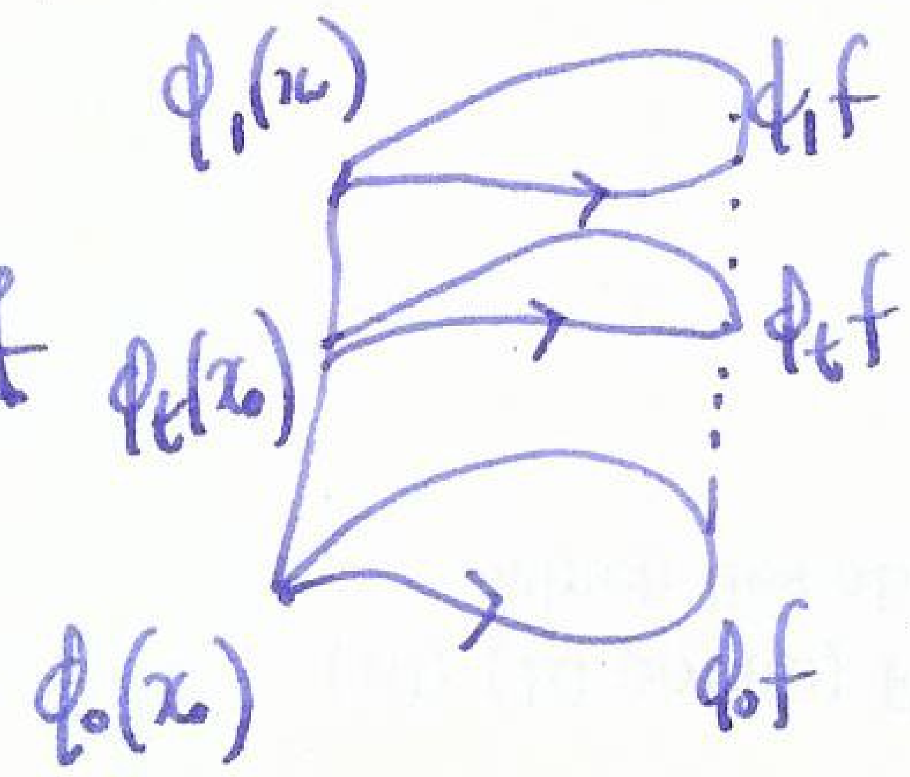
Lemma If $\phi_t:X\to Y$ is a homotopy and h is the path $h(t)=\phi_t(x_0)$

then

$$\begin{tikzcd}
 & \pi_1(Y,\phi_1(x_0)) \arrow[d, "\beta_h"] \\
\pi_1(X,x_0) \arrow[ur, "(\phi_1)_*"] \arrow[r, "(\phi_0)_*"'] & \pi_1(Y,\phi_0(x_0))
\end{tikzcd}$$

commutes, i.e. $(\phi_0)_*=\beta_h(\phi_1)_*$ where β_h is the change of basepoint map.

Proof



let $h_t(s)=h(ts)$

(i.e. $h_t:[0,t]\xrightarrow{\text{stretch}}[0,1]\to Y$.

let $f:I\to X$ be a loop in X

then $h_t\cdot\phi_t f\cdot\bar h_t$ is a homotopy of loops based at $\phi_0(x_0)$.

so $h_0\cdot\phi_0 f\cdot\bar h_0\simeq\underbrace{h_1}_{=h}\cdot\phi_1 f\cdot\underbrace{\bar h_1}_{=\bar h}\Rightarrow(\phi_0)_*[f]=\beta_h(\phi_1)_*[f]$. □

<u>Proof</u> (of Prop^{n}) $\quad \phi: X \underset{\psi}{\rightleftarrows} Y$ homotopy equivalence, so let $\psi: Y \to X$

s.t. $\phi\psi \simeq id_Y \quad \psi\phi \simeq id_X$

consider

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \phi\psi\phi(x_0))$$

$$\psi\phi \simeq id_X \Rightarrow \psi_*\phi_* = \underbrace{\beta_h \, \mathbb{1}}_{\text{isomorphism}} \quad \Leftarrow \text{isomorphism}$$

$\Big\} \Rightarrow \phi_*$ injective, ψ_* surjective

similarly $\phi_*\psi_*$ is an isomorphism $\Big\} \Rightarrow \psi_*$ injective, ϕ_* surjective $\quad$ so ϕ_* isomorphism $\square$.

§1.2 Van Kampen's Theorem

<u>Free products</u>

<u>Example</u> $F_2 = \langle a, b \mid \rangle$ $\quad$ Set W = all finite strings of $\{a, a^{-1}, b, b^{-1}\}$.

e.g. $\phi \in W$. $\quad a \in W \quad abab^{-1} \in W$.
$aabb^{-1}a \in W$.

define an equivalence relation $\sim$ on W by

$$uaa^{-1}v \sim uv \qquad u, v \in W.$$
$$ua^{-1}av \sim uv$$
$$ubb^{-1}v \sim uv$$
$$ub^{-1}bv \sim uv$$

notation: $\underbrace{aa\cdots a}_{} = a^n$.

$W/\sim$ may be identified with the set of <u>reduced words</u> (minimal elements of equivalence classes)

<u>multiplication</u> given by concatenation i.e. $u \cdot v = uv$

e.g. $(aba) \cdot (a^{-1}b^{-1}aa) = ab\cancel{a}\cancel{a^{-1}}\cancel{b^{-1}}aa = a^3$.

<u>Exercise</u> this defines a group called the <u>free group on 2 generators</u> F_2.
(identity element $= \phi = 1$).

Generalizations · F_n free group on n-generators, i.e. W = all finite strings of $\{a_1, a_1^{-1}, a_2, a_2^{-1}, \dots a_n, a_n^{-1}\}$.

· Free products: Let G, H be groups, the free product $G * H$ consists of all strings/words of the form $a_1 a_2 a_3 \dots a_n$ $a_i \in G$ or H up to the following equivalences: · if $a_i a_{i+1}$ both in same group

then $u a_i a_{i+1} v = a (a_i a_{i+1}) v$

· 1 in any gp corresponds to ϕ

a minimal length representative of an equivalence class is called a reduced word.

$G * H$ is a group · multiplication is concatenation

$$(a_1 a_2 \dots a_k)(b_1 b_2 \dots b_\ell) = a_1 a_2 \dots a_k b_1 b_2 \dots b_\ell$$ ← may not be reduced!

· identity is ϕ

· inverses: $(a_1 a_2 \dots a_k)^{-1} = a_k^{-1} a_{k-1}^{-1} \dots a_1^{-1}$

Examples

- $\mathbb{Z} * \mathbb{Z} \cong F_2$
- $\mathbb{Z}_2 * \mathbb{Z}_2$ a free product which is not a free group.

Universal property $\ast_\alpha G_\alpha$ (arbitrary) free product

any collection of homomorphisms $\phi_\alpha : G_\alpha \to H$ extends to a unique homomorphism $\phi : \ast_\alpha G \to H$.

Group presentations $G = \langle a_1, a_2, \dots a_n \mid r_1, r_2, \dots r_m \rangle$

generators ↑ relations (i.e. elements of F_n).

$G = F_n / N$ $\quad F_n$ = free group generated by $a_1, \dots a_n$
N = normal subgroup generated by $r_1 \dots r_m$.

Example $F_2 = \langle a, b \mid \rangle$. $\quad \mathbb{Z} = \langle a \mid \rangle$.

$\mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle = \langle a, b \mid aba^{-1}b^{-1} \rangle$.

$\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = 1, b^2 = 1 \rangle$.

Warning Given a group presentation, there is no algorithm

- to decide if the group is trivial
- to decide if a given word is trivial.

Abelianization For any group G, there is an abelian group $ab(G)$ s.t. any ~~map~~ homomorphism $\phi : G \to A$ (abelian) factors through $ab(G)$.

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "ab"] & A \\ ab(G) \arrow[ur, "\phi'"] & \end{tikzcd}$$

Given a presentation of G, you can find $ab(G)$ by abelianizing the presentation

e.g. $F_2 = \langle a, b \mid \rangle \quad ab(F_2) = \langle a, b \mid ab = ba \rangle. \cong \mathbb{Z}^2$.

Useful fact $ab(G) \ncong ab(H) \Rightarrow G \ncong H$.

Corollary $F_n \ncong F_m$ for $m \neq n$.

Warning: · knowing a presentation for G doesn't necessarily help you know which group G is.

· not nec. easy to write down presentations for Lie groups. e.g $PSL(2, \mathbb{Z})$?

in fact $PSL(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. gen by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.