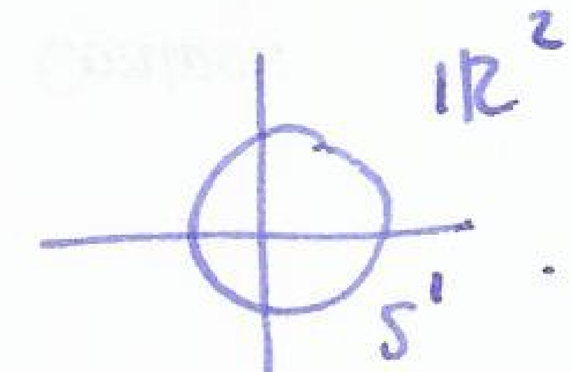
Thm If $f: S^2 \to \mathbb{R}^2$ cts, then there is a pair of antipodal points $x, -x$ s.t. $f(x) = f(-x)$ [Borsuk-Ulam in dim 2]

Corollary not injective cts map from $S^2 \to \mathbb{R}^2$.

Proof spose not, $f: S^2 \to \mathbb{R}^2$ $f(x) \neq f(-x)$ for all x.

define $g: S^2 \to S^1$ by $g(x) = \dfrac{f(x) - f(-x)}{|f(x) - f(-x)|}$



let $\alpha: I \to S^2$ be the loop $\alpha(s) = (\cos(2\pi s), \sin(2\pi s), 0)$

then $h = g\alpha: I \to S^1$ loop

note: $g(-x) = -g(x)$ so $h(s+\frac{1}{2}) = -h(s)$ on $s \in [0, \frac{1}{2}]$

lift to $\tilde{h}: [0, \frac{1}{2}] \to \mathbb{R}$ so $\tilde{h}(s+\frac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$ q odd ~~(may depend on s)~~

q doesn't depend on s: as $q = 2\tilde{h}(s+\frac{1}{2}) - \tilde{h}(s)$ cts on $[0, \frac{1}{2}]$.

so $\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q$ (odd), so $[h] = q \neq 0 \in \pi_1(S^1) \cong \mathbb{Z}$.

so h not null homotopic. but α null homotopic in S^2 ⨳.

Corollary whenever S^2 is the union of 3 closed sets A_1, A_2, A_3, then at least one contains a pair of antipodal points.

Proof let $d_i: S^2 \to \mathbb{R}$ be distance to A_i $d_i(x) = \inf_{y \in A_i} |x - y|$ cts.

apply Borsuk-Ulam to $S^2 \to \mathbb{R}^2$, $x \mapsto (d_1(x), d_2(x))$

gives x with $\left.\begin{array}{l} d_1(x) = d_1(-x) \\ d_2(x) = d_2(-x) \end{array}\right\}$ if either zero then $\pm x \in A_1$ or $\pm x \in A_2$; if neither zero then $\pm x \in A_3$. $\square$

π_1 of a product.

<u>Prop^n</u> $\pi_1(X \times Y) \cong \pi_1 X \times \pi_1 Y$. if X, Y path connected.

<u>Proof</u> recall product topology $f: Z \to X \times Y$ cts iff $g: Z \to X$ cts
$h: Z \to Y$
where $f(z) = (g(z), h(z))$ $(g = \pi_X f, \ h = \pi_Y f)$

a loop $f: I \to X \times Y$ is therefore a pair of loops $g: I \to X$
$h: I \to Y$

similarly a homotopy $f_t: I \to X \times Y$ gives a pair of homotopies $g_t: I \to X$
$h_t: I \to Y$.

this gives a bijection $\pi_1(X \times Y, (x_0, y_0)) \overset{\phi}{\longleftrightarrow} \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

this is a group homomorphism: $f_1 \cdot f_2 = (g_1, h_1) \cdot (g_2, h_2)$.
$= (g_1 \cdot g_2, h_1 \cdot h_2)$.

so an isomorphism $\square$.

<u>Example</u> torus $T = S^1 \times S^1$ $\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$

n-torus $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}}$ $\pi_1(T^n) \cong \mathbb{Z}^n$.

<u>Induced homomorphisms</u>

Let $\phi: X \to Y$ cts s.t. $\phi(x_0) = y_0$

<u>claim</u> there is an induced homomorphism $\phi_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$
given by $[f] \mapsto [\phi \circ f]$

<u>Proof</u> check well defined: suppose $f_0 \simeq f_1$ by f_t then $\phi f_0 \simeq \phi f_1$ by ϕf_t.

· homomorphism $\phi(f \cdot g)$ and $\phi f \cdot \phi g$ are equal as <u>functions</u>.

$$= \begin{cases} \phi f(2s) & 0 \le s \le \frac{1}{2} \\ \phi g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases} \quad \square$$

Useful properties
- $(X,x_0) \xrightarrow{\psi} (Y,y_0) \xrightarrow{\phi} (Z,z_0)$ then $(\phi\psi)_* = \phi_*\psi_*$
- $(X,x_0) \underset{id_X}{\xrightarrow{\mathbb{1}_X}} (X,x_0)$ $\quad (\mathbb{1}_X)_* = \mathbb{1} = id_{\pi_1(X,x_0)}$.

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Proof:
- associativity $\phi(\psi f) = (\phi\psi)f$.
- $f \mapsto f$ induces an iso $[f] \to [f]$ in $\pi_1(X,x_0)$. $\square$.

Remark: if X, Y homeomorphic $\quad (X,x_0) \underset{\text{inverse}}{\overset{\phi}{\rightleftarrows}} (Y,y_0)$

$\phi^{-1} = \psi$.

then $\psi\phi = id_X \quad \phi\psi = id_Y$.

so $(\psi\phi)_* = id_{\pi_1(X,x_0)} \quad (\phi\psi)_* = id_{\pi_1(Y,y_0)} \Rightarrow \psi_*, \phi_*$ isomorphisms.

so $\pi_1(X,x_0) \cong \pi_1(Y,y_0)$.

Prop^n: $\pi_1(S^n) = 0$ if $n \geq 2$.

Proof Let $f: I \to S^n$ be a loop at x_0. If $f(I)$ disjoint from some point $x \in S^n$

then f is a loop in $S^n \setminus \{x\} \cong \mathbb{R}^n$, simply connected, so f null homotopic.

claim: we can homotope f to be non-surjective.

proof (of claim): Let B be a small open ball around x. ($x_0 \notin B$)

consider segments of $f(I)$ which enter B, hit x and leave B.

$f^{-1}(B)$ open in $[0,1]$ consists of (arbitrary) union of open intervals.

$f^{-1}(x)$ closed $\Rightarrow$ compact in $[0,1]$, connected components of $f^{-1}(B)$ open cover, so has finite subcover

$(a_i, b_i)_{i=1,\dots,N}$. for $f: (a_i,b_i) \to B$ note $f(a_i) \in \partial B$, $f(b_i) \in \partial B$

so choose $g: (a_i,b_i) \to \partial B$ for $g(a_i) = f(a_i)$, $g(b_i) = f(b_i)$

$f|_{(a_i,b_i)} \simeq g|_{(a_i,b_i)}$ by straight line homotopy. as B homeo to subset of $\mathbb{R}^n$.

so $f_0 = f$ homotopic to $f_1 = \begin{cases} f & \text{on } I \setminus \bigcup (a_i,b_i) \\ g_i & \text{on } (a_i,b_i) \end{cases}$ $\quad f_1$ avoids x. $\square$.

Application $\mathbb{R}^n \setminus \{x\} \simeq S^{n-1}$ so $\pi_1(\mathbb{R}^n \setminus \{x\}) = \mathbb{Z}$ $n=2$
$= 1$ $n \neq 2$

Corollary $\mathbb{R}^2$ not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.

Induced maps

Prop^n If $r: X \to A$ is a retraction (recall $i: A \subset X$ and $r|_A = id_A$).
then $i_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$ is injective.
If A is a deformation retraction, then i_* is an isomorphism.

Def^n $r_t: X \to X$ is a deformation retraction to A if r_t ~~is~~ homotopy from
$r_0 = 1_X$ to $r_1: X \to A$ retraction.

Proof $A \xhookrightarrow{i} X \xrightarrow{r} A$ $\quad ri = 1_A$
induces $\pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X) \xrightarrow{r_*} \pi_1(A)$ so $r_* i_* = 1_{\pi_1(A)} \Rightarrow i_*$ injective
r_* surjective.

now suppose $r_t: X \to X$ deformation retraction.
let $f: I \to X$ be a loop, then $r_t f: I \to X$ is a homotopy that
takes $r_0 f = 1_X f = f$ to $r_1 f = rf \subset A$ so $[rf] \in \pi_1(A)$
is homotopic to $[f] \in \pi_1(X)$ so $i_*: \pi_1(A) \to \pi_1(X)$ surjective. $\square$.

Th^m (Brouwer fixed point th^m) $S^1 = \partial D^2$ is not a retract of D^2.

Proof suppose there is a retract. $S^1 \xhookrightarrow{i} D^2 \xrightarrow{r} S^1$ s.t. $r_* i_* = id_{\pi_1 S^1}$.
$\pi_1 S^1 \xrightarrow{i_*} \pi_1 D^2 \xrightarrow{r_*} \pi_1 S^1$
$\mathbb{Z} \to 0 \to \mathbb{Z}$ not injective. # $\square$

Pairs of spaces: (X, A) means $A \subset X$.
$f: (X, A) \to (Y, B)$ cts. means $f: X \to Y$ cts
and $f(A) \subset f(B)$

special case $f: (X, x_0) \to (Y, y_0)$ a map of pairs is called <u>basepoint preserving</u>
i.e. $f(x_0) = y_0$