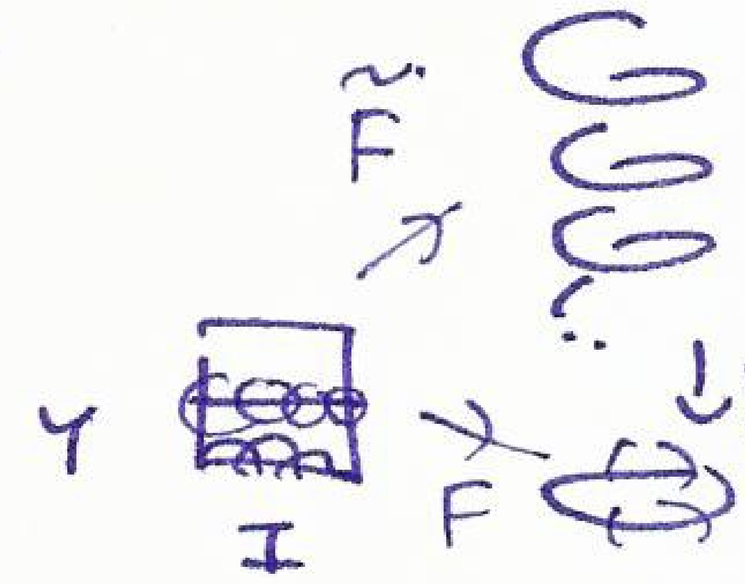
<u>special property</u> of $p: S^1 \to \mathbb{R}$

(*) there is an open cover $\{U_\alpha\}$ of S^1 such that for each U_α, $p^{-1}(U_\alpha)$ is a disjoint union of sets each of which is homeomorphic to U_α by p.

<u>Example</u> choose $S^1 = [0,2\pi]/\sim$ $\quad U_1 = (\frac{\pi}{4}, \frac{7\pi}{8})$, $\quad U_2 = (-\frac{3\pi}{4}, \frac{3\pi}{4})$.

<u>Proof</u> (of c) using (*)) $F: Y \times I \to S^1$, $\tilde{F}: Y \times \{0\} \to \mathbb{R}$



F cts, so each $(y_0, t) \in Y \times I$ has a product nbd $N_t \times (a_t, b_t)$ s.t. $F(N_t \times (a_t, b_t)) \subset U_\alpha$ for some single $U_\alpha \subset S^1$.

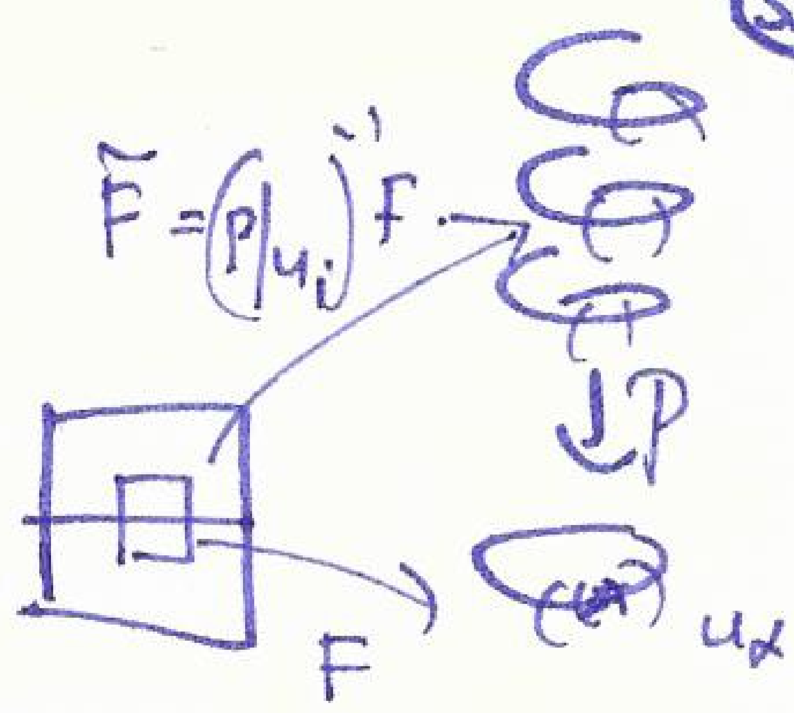
$\{y_0\} \times I$ compact, so there is a finite cover.

Choose a nbd N of y_0, and a partition $0 \le t_0 < t_1 < \dots < t_n = 1$ of $[0,1]$ such that for each i, $F(N \times (t_i, t_{i+1})) \subset U_i$ for some U_i.

Assume $\tilde{F}$ has been constructed on $N \times [0, t_i]$, then $F(N \times [t_i, t_{i+1}]) \subset U_i$, so by (*) there is an open set $\tilde{U}_i \subset \mathbb{R}$ s.t. $p|_{\tilde{U}_i}: \tilde{U}_i \to U_i$ is a homeomorphism, and $\tilde{U}_i$ contains $\tilde{F}((y_0, t_i))$.

Now define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ by $\tilde{F} = (p|_{U_i})^{-1} F$

Repeat finitely many times to construct $\tilde{F}$ on $N \times I$.



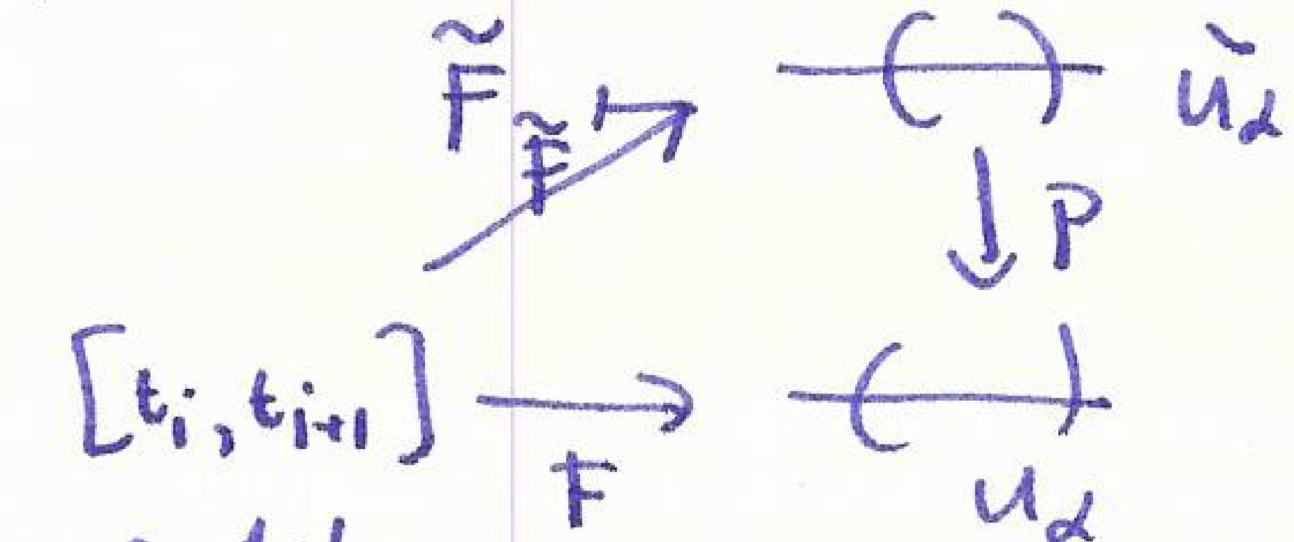
<u>uniqueness</u>: special case $Y = \{pt\}$

suppose $\tilde{F}, \tilde{F}'$ are two lifts $F : I \to S^1$

such that $\tilde{F}(0) = \tilde{F}'(0)$.

induct on length of partition $0 \leq t_0 < t_1 < \cdots < t_n = 1$

so $\tilde{F}(t_i) = \tilde{F}'(t_i)$



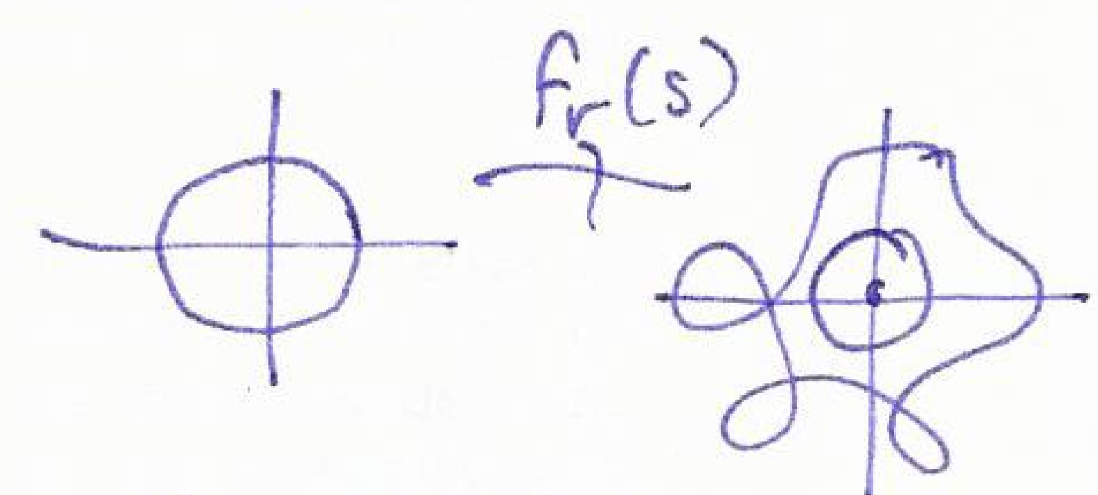
but then

$p|_{\tilde{u}_\alpha}$ homeo $\Rightarrow$ $p\tilde{F} = p\tilde{F}' = F$ $\Rightarrow$ $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$.

<u>Finally</u>: uniqueness on $\{y\} \times I$ $\Rightarrow$ if N, N' overlap then $\tilde{F}|_N = \tilde{F}|_{N'}$ on intersection, so gives lift $\tilde{F} : Y \times I \to \mathbb{R}$. $\square$

<u>Applications</u>

<u>Thm</u> (Fundamental theorem of algebra) Every nonconstant polynomial with coeffs in $\mathbb{C}$ has a root in $\mathbb{C}$.



<u>Proof</u> $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$

If $p(z)$ has no roots in $\mathbb{C}$, then for each $r \in \mathbb{R}$, $r \geq 0$

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$ defines a loop in S^1 based at 1.

f_r is a homotopy of loops based at 1 (ck!).

f_0 is the trivial loop $f_0(s) = 1$.

so $[f_r] \in \pi_1 S^1 \cong \cdot\ [f_r] = 0$ for all r.

pick $r \gg 1 + |a_1| + \cdots + |a_n|$

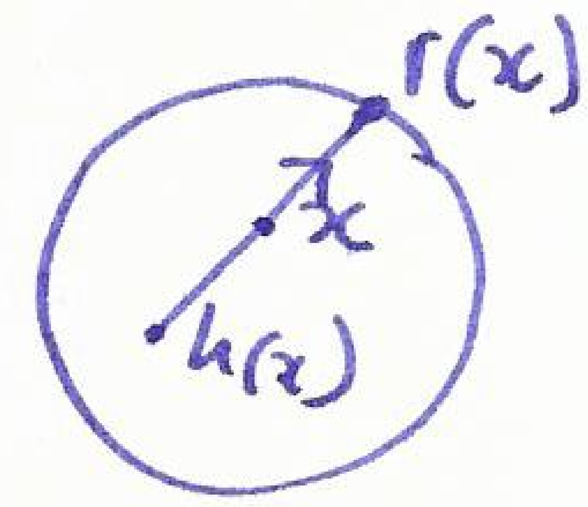
then $|z^n| = r^n = r \cdot r^{n-1} > (|a_1| + \cdots + |a_n|)\,|z^{n-1}| \geq |a_1 z^{n-1} + \cdots + a_n|$

consider $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$, then this has no roots on the circle $|z| = r$, and for $0 \leq t \leq 1$

$f_r \simeq p_t \simeq p_0$ but $p_0(z) = z^n = [\omega_n] \neq [\omega_0]$ #.

(Brouwer fixed pt thm special case).

Thm Every cts map $h: D^2 \to D^2$ has a fixed point, i.e. there is an $x \in D^2$ s.t. $h(x) = x$.



Proof spose $h(x) \neq x$ for all $x \in D^2$

define $r: D^2 \to S^1$ by setting $r(x)$ to be the intersection of the ray from $h(x)$ thru x with $\partial D^2 = S^1$.

r is a retraction i.e. $r: D^2 \to S^1$ cts and $r(x) = x$ for all $x \in S^1$.

claim: there is no retraction $r: D^2 \to S^1$.

Let f_0 be any loop in S^1. In D^2 f_0 is homotopic, by f_t, to the constant loop, e.g. by linear homotopy. But then rf_t is a homotopy from f_0 to constant loop in S^1. #.