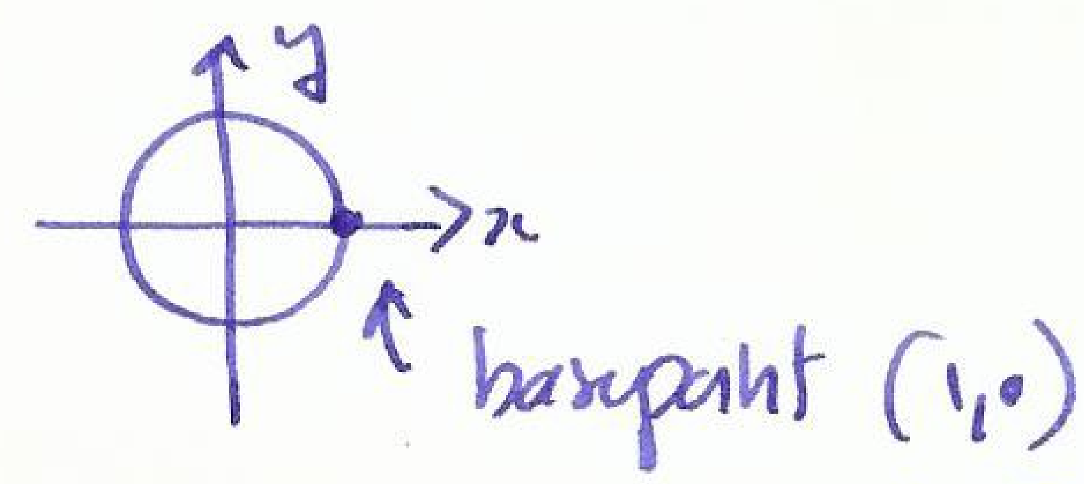
Warning: connected $\not\Rightarrow$ path connected.

Example

Fundamental group of the circle

S^1 : $(x,y) \in \mathbb{R}^2$ such that $x^2+y^2=1$.

$z \in \mathbb{C}$ $\quad |z| = 1$.



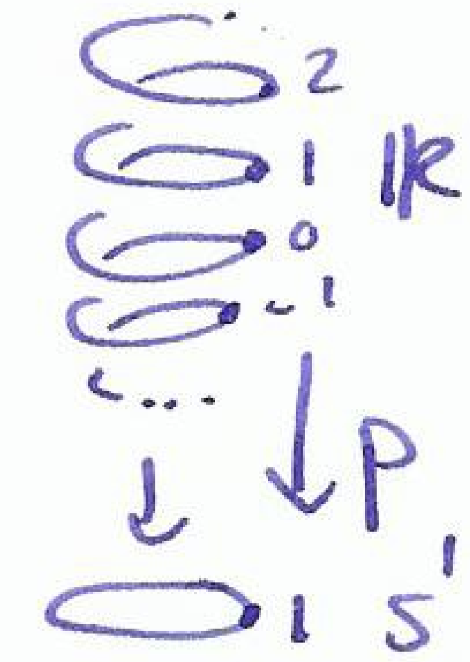
$1 \in \mathbb{C}$.

examples of loops: $\omega_n(s) = (\cos(2\pi n s), \sin(2\pi n s))$

$\theta \longmapsto e^{2\pi i \theta n}$.

Thm The map $\Phi : \mathbb{Z} \to \pi_1(S^1)$ is an isomorphism.

$n \longmapsto [\omega_n(s)]$



Proof There is a map $p : \mathbb{R} \to S^1$

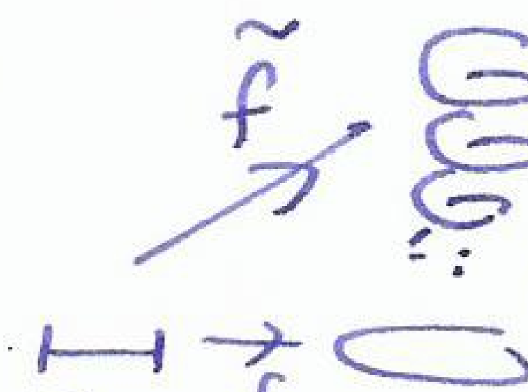
$t \mapsto e^{2\pi i t}$

nb $\omega_n(s) = p\tilde{\omega}_n$ where $\tilde{\omega}_n : I \to \mathbb{R}$

$[0,1] \to [0,n]$

$s \longmapsto ns$

Defn if $f : I \to S^1$ is a path, and $\tilde{f} : I \to \mathbb{R}$ is a path such that $f = p\tilde{f}$, then we say $\tilde{f}$ is a lift of f.



Note we could define $\Phi(n)$ to be $p\tilde{f}$ for $\tilde{f} : I \to \mathbb{R}$ any path from 0 to n, as all such paths are homotopic by the linear homotopy.

check: Φ is a homomorphism (i.e. $\Phi(m+n) = \Phi(m) \cdot \Phi(n)$).

notation: Let $\tau_m : \mathbb{R} \to \mathbb{R}$ be the translation $\tau_m(x) = x+m$

consider $\underbrace{\tilde{\omega}_m}_{\text{path from } 0 \text{ to } m} \cdot \underbrace{\tau_m \tilde{\omega}_n}_{\text{path from } m \text{ to } m+n} \simeq \underbrace{\tilde{\omega}_{m+n}}_{\text{path from } 0 \text{ to } m+n.}$

therefore

$$
\begin{array}{ccc}
p(\tilde{\omega}_m \cdot \tau_m \tilde{\omega}_n) & \simeq & p\tilde{\omega}_{m+n} \\
\| & & \| \\
p\tilde{\omega}_m \cdot p\tau_m \tilde{\omega}_n & & \Phi(m+n) \\
\| & & \\
p\tilde{\omega}_m \cdot p\tilde{\omega}_n & & \\
\| & & \\
\Phi(m) \cdot \Phi(n) & & \text{as required. } \square .
\end{array}
$$

<u>Φ is an isomorphism</u>

<u>useful facts</u>

a) for each path $f : I \to S^1$ starting at $x_0 \in S^1$, and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \to \mathbb{R}$ starting at $\tilde{x}_0$

b) for each homotopy $f_t : I \to S^1$ of paths starting at x_0, and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lifted homotopy $\tilde{f}_t : I \to \mathbb{R}$ of paths starting at $\tilde{x}_0$.

<u>Claim</u> a), b) $\Rightarrow$ <u>Thm</u>.

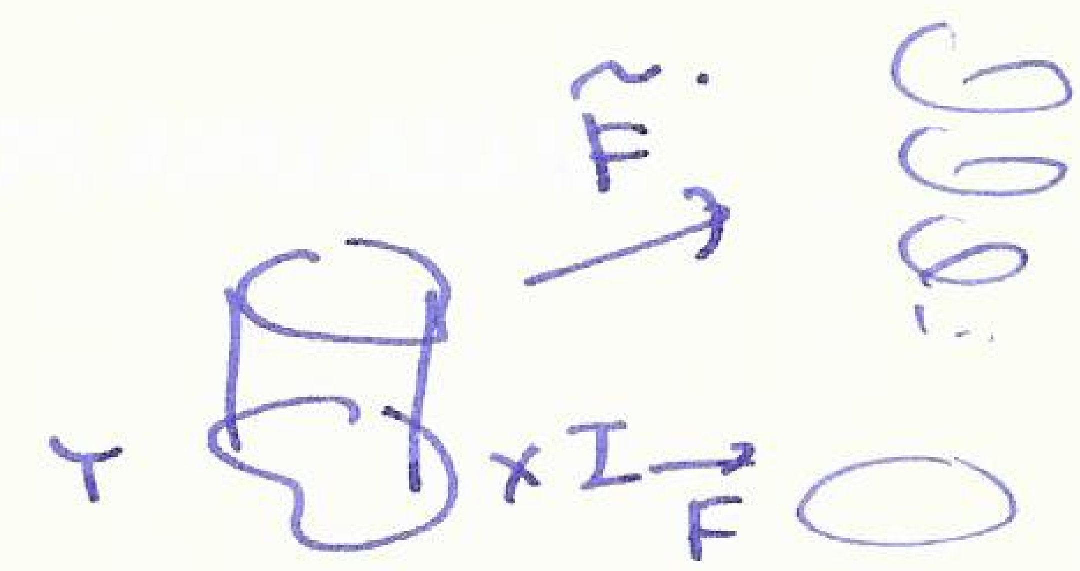
<u>Proof</u> (of claim)

- <u>Φ surjective</u> : let $f : I \to S^1$ be a loop based at $x_0 = (1,0)$ representing an element of $\pi_1(S^1)$. a) $\Rightarrow$ there is a lift $\tilde{f} : I \to \mathbb{R}$ starting at $0 \in \mathbb{R}$. As $p\tilde{f}(1) = x_0$, this implies $\tilde{f}(1) \in p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$, so $\tilde{f}$ is a path in $\mathbb{R}$ from 0 to $n \in \mathbb{Z}$, so is homotopic to $\tilde{\omega}_n$ for some n. So $\tilde{f} \simeq \tilde{\omega}_n \Rightarrow p\tilde{f} \simeq \omega_n \Rightarrow [f] = [\omega_n] = \Phi(n)$.

• Φ injective : suppose $\Phi(m) = \Phi(n)$, i.e. $\omega_m \simeq \omega_n$.

Let f_t be a homotopy from $\omega_m = f_0$ to $\omega_n = f_1$. By b) this lifts to a homotopy of paths starting at $0 \in \mathbb{R}$. Uniqueness $\Rightarrow$ $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$. As $p\tilde{f}_t(1) = x_0 = (1,0)$ for all t, so $\tilde{f}_t(1) \in p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$ for all t. $\tilde{f}_t$ is map $I \to \mathbb{Z}$ so constant,

so $f_0(1) = f_1(1) \Rightarrow m = n$ $\square$.



Proof (of a), b)) we'll prove c):

c) given $F: Y \times I \longrightarrow S^1$ and a map $\tilde{F}: Y \times \{0\} \to \mathbb{R}$ lifting $F|_{Y\times\{0\}}$ then there is a unique map $\tilde{F}: Y \times I \to \mathbb{R}$ lifting F, and restricting to $\tilde{F}$ on $Y \times \{0\}$.

c) $\Rightarrow$ a) : choose $Y = \{pt\}$

c) $\Rightarrow$ b) : $f_t(s) \leftrightarrow F: I \times I \longrightarrow S^1$

$$F(s,t) = f_t(s)$$

a) implies there is a unique lift $\tilde{F}: I \times \{0\} \to \mathbb{R}$, then c) implies there is a unique lift $\tilde{F}: I \times I \to \mathbb{R}$

note: $\tilde{F}|_{\{0\}\times I}$ and $\tilde{F}|_{\{1\}\times I}$ are lifts of the constant path, and so are constant, so $\tilde{F}$ is a path homotopy.