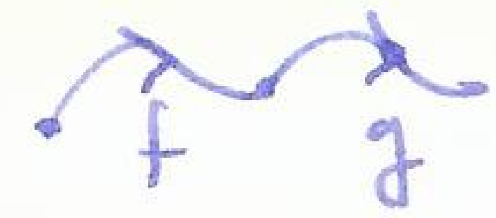
composition of paths: spse $f: I \to X$, $g: I \to X$ s.t. $f(1) = g(0)$



then there is a path $f \cdot g : I \to X$ defined by

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

Claim: composition respects homotopy classes, i.e. if $f_0 \simeq f_1$ and $g_0 \simeq g_1$, the $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ ($f_0(1) = g_0(0)$).

Proof given homotopies f_t and g_t, define

$$h_t = f_t \cdot g_t = f_t \cdot g_t(s) = \begin{cases} f_t(2s) & 0 \le s \le \frac{1}{2} \\ g_t(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

is a homotopy from $f_0 \cdot g_0$ to $f_1 \cdot g_1$.

Defn A loop based at x_0 is a path $f: I \to X$ which starts and ends at x_0.

Notation: the set of all homotopy classes of loops based at x_0 is written $\pi_1(X, x_0)$.

Propn $\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \cdot g]$

This is called the fundamental group of X with basepoint x_0.

Proof note: • $f \cdot g$ makes sense as $f(1) = x_0 = g(0)$.

• $[f][g] = [f \cdot g]$ is well defined as compositions of homotopic paths are homotopic

check: group axioms.

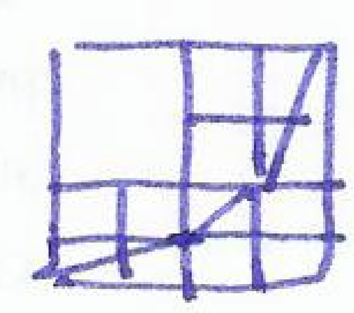
Defn A reparameterization of a path $f: I \to X$ is a composition $f\phi$ where $\phi: I \to I$ is any continuous map s.t. $\phi(0) = 0$ and $f(1) = 1$.

Note: all reparameterizations are homotopic via $f\phi_t(s) = (1-t)\phi(s) + ts$

Associativity: $(f \cdot g) \cdot h$ is a reparameterization of $f \cdot (g \cdot h)$ by

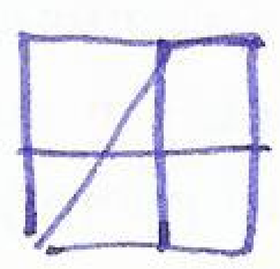
f g h

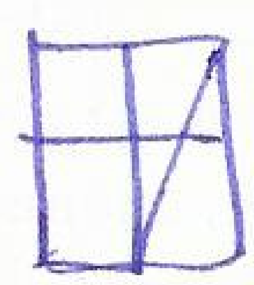
f g h



so $(f \cdot g) \cdot h$ homotopic to $f \cdot (g \cdot h)$ so $([f][g])[h] = [f]([g][h])$

Identity let $c: I \to X$ be the constant path $c(t) = x_0$ then $f \cdot c$, $c \cdot f$, f are all reparameterizations of f by





so $[c]$ is a two sided identity in $\pi_1(X, x_0)$

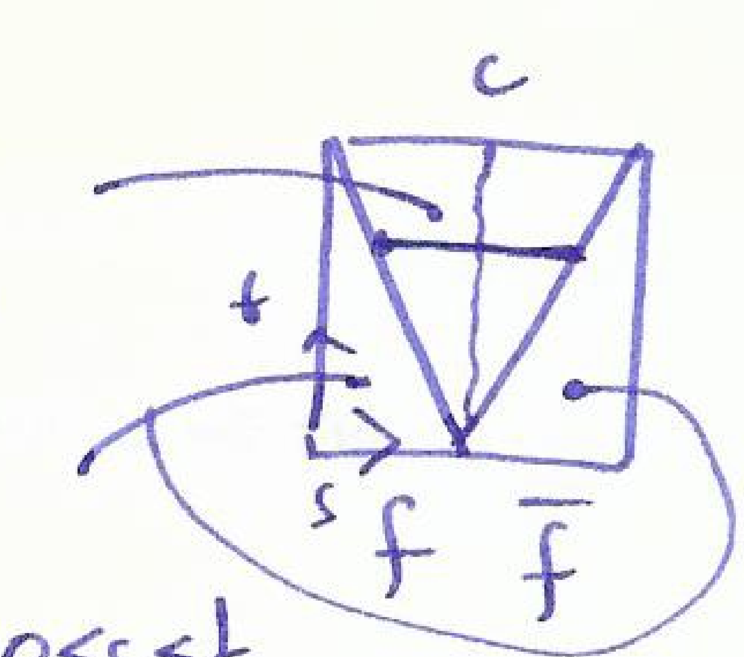
Inverses let $\bar{f}(t) = f(1-t)$. claim: $f \cdot \bar{f}$ homotopic to c

proof: use homotopy $h_t = f_t \cdot \bar{f}_t$ where $f_t = \begin{cases} f(s) & 0 \le s \le 1-t \\ f(1-t) & s > 1-t \end{cases}$

and $g\bar{f}_t(s) = \begin{cases} f(1-t) & s<t \\ f(1-s) & t<s\le 1 \end{cases}$

$g_t = \overline{(f_t)}$

$H(s,t) = f(1-t)$



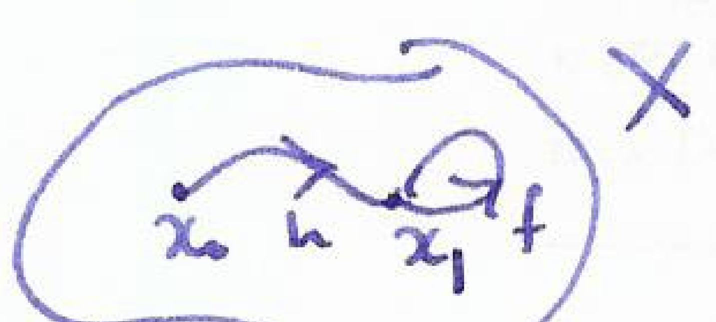
$H(s,t) = \begin{cases} f(2s) & 0\le s\le \frac{1}{2} \\ f(2-2s) & \frac{1}{2}\le s\le 1 \end{cases}$

so $[\bar{f}]$ is a two sided inverse for $[f]$.

<u>Example</u> $X \subset \mathbb{R}^n$ convex, then $\pi_1(X, x_0) = 0$ trivial group.

any two loops $f, g: I \to X$ homotopic to ~~a constant map~~ each other by the linear homotopy $f_t(s) = (1-t)f(s) + tg(s)$.

<u>Change of basepoint</u>



let $h: I \to X$ be a path from x_0 to x_1

if f is a loop at x_1, then $h\cdot f\cdot \bar{h}$ is a loop at x_0.

<u>Prop^{n}</u> $\beta_h: \pi_1(X, x_1) \to \pi_1(X, x_0)$ given by $\beta_h[f] = [h\cdot f\cdot \bar{h}]$ is an isomorphism.

<u>Proof</u> · <u>well defined</u>: suppose $f_0 \simeq f_1$ by f_t. then $h\cdot f_t\cdot \bar{h}$ is a homotopy from $h\cdot f_0\cdot \bar{h}$ to $h\cdot f_1\cdot \bar{h}$.

· <u>homomorphism</u>: $\beta_h[f\cdot g] = [h\cdot f\cdot g\cdot \bar{h}] = [h\cdot f\cdot \bar{h}\cdot h\cdot g\cdot h] = [h\cdot f\cdot \bar{h}][h\cdot g\cdot h]$

$= \beta_h[f]\beta_h[g]$.

<u>isomorphism</u>: consider $\beta_{\bar{h}}: \pi_1(X, x_0) \to \pi_1(X, x_1)$

$\beta_{\bar{h}}\beta_h([f]) = \beta_{\bar{h}}[h\cdot f\cdot \bar{h}] = [\bar{h}\cdot h\cdot f\cdot \bar{h}\cdot h] = [f]$ so $\beta_{\bar{h}} = \beta_h^{-1}$. D.

<u>Def^{n}</u> X is <u>simply connected</u> if it is path connected and has trivial fundamental group.

<u>Def^{n}</u> X is <u>path connected</u> if for any two $x, y \in X$ there is a path $f: I \to X$ s.t. $f(0) = x$ and $f(1) = y$.

<u>Prop^{n}</u> path connectedness is an equivalence relation on points of X. classes called <u>path components</u>.