Lebesgue numbers for covers.

Def<u>n</u> X metric space $A \subset X$, $\{U_i\}$ open cover for A. $\delta > 0$ is a <u>Lebesgue number</u> for $\{U_i\}$ if forall $a \in A$ every $B(a, \delta) \subset U_i$ for some single $U_i \in \{U_i\}$.

Lemma Every open cover of a sequentially compact subset of a metric space has a Lebesgue number.

Proof $A \subset X$ metric space, A seq. compact. Let $\{U_i\}$ be an open cover of A. Suppose no Lebesgue number, then for every $\delta > 0$ there is an $B(a_\delta, \epsilon)$ not contained in a single U_i. $(\delta = \frac{1}{n})$ a_n has a convergent subsequence $a_n' \to a \in A$. But then $a \in U_i$ for some i so as U_i open $\exists\ \epsilon > 0$ with $a \in B(a, \epsilon) \subset U_i$ #.

Thm X metric space: compact ① $\Leftrightarrow$ countably compact ② $\Leftrightarrow$ sequentially compact ③.

Proof ① $\Rightarrow$ ② spose $\{a_i\}$ has no accumulation point, then for every $p \in X$ there is an open set U_p s.t. $U_p \cap \{a_i\}$ is finite. The U_p form an open cover of X, so there is a finite cover, $\Rightarrow$ $\{a_i\}$ is finite $\Rightarrow$ sequence has constant subsequence, so has accumulation point. $\square$.

② $\Rightarrow$ ③ Spose $\{a_n\}$ has an accumulation point p, then every $B(p, \frac{1}{n})$ contains some $a_{m(n)}$, then subsequence $a_{m(n)} \to p$. $\square$.

③ $\Rightarrow$ ① recall: sequential compactness $\Rightarrow$ • Lebesgue number for an open cover
$\Rightarrow$ • X totally bounded.

Let $\{U_i\}$ be an open cover with Lebesgue number δ. Totally bounded means for any $\epsilon > 0$ there is a finite set $\{a_1, \dots a_n\}$ with $X \subset \bigcup B(a_i, \epsilon)$. Choose $\epsilon = \delta$, then $X \subset \bigcup B(a_i, \delta)$ for some $\{a_1, \dots a_n\}$ finite. But each $B(a_i, \delta) \subset U_{j(i)}$ for some $j(i)$, so $\{U_{j(i)}\}$ is a finite subcover. $\square$.

§12 Product spaces

(X_i, T_i) topological spaces. $X = \prod_i X_i$ product set. projection $\pi : X \to X_i$

Def<u>n</u> The product topology (or Tychonoff topology) is the coarsest topology on $X = \prod X_i$ such that every projection is continuous.

Base <u>finite products</u>: $X_1 \times \cdots \times X_n$ X_i has base B_i, then $B = \{U_1 \times \cdots \times U_n \mid U_i \in B_i\}$ is a ~~subbase~~ base for the product topology on X

<u>infinite products</u>: $X = \prod X_i$ B_i base for X_i, then $B = \{X_1 \times \cdots \times X_{n-1} \times U_n \times X_{n+1} \times \cdots \mid U_n \in B_n\}$ is a <u>sub</u>base for the product topology. (intersections called defining base / product base).

<u>Warning</u>: $U_1 \times U_2 \times U_3 \times \ldots$ U_i open in X_i is not necessarily open in X!

<u>Thm</u> (Tychonoff) The product of compact spaces is compact $\square$

<u>Useful facts</u>: • $f: Y \xrightarrow{f} X = \prod X_i$ is continuous iff $\pi_i \circ f$ is continuous for every X_i

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "\pi_i \circ f"'] & X \arrow[d, "\pi_i"] \\ & X_i \end{tikzcd}$$

observation

• defining base set $B = X_1 \times \cdots \times U_{k_1} \times \cdots \times U_{k_n} \times X_{k_n+1} \times \cdots$ (finitely many U_{k_i})

then $\pi_i(B)$ open for each i.

• projection map $\pi_i : X \to X_i$ is an <u>open</u> map.

§13 Connectedness

<u>Defn</u> X is <u>disconnected</u> if there are disjoint open sets U, V with $X = A \cup B$.

<u>Defn</u> $A \subset X$ is <u>disconnected</u> if it is disconnected in the relative topology

i.e. there are disjoint open set U, V in X s.t. $A \subset U \cup V$ and $A \cap U \neq \phi$, $A \cap V \neq \phi$.

otherwise X or A is <u>connected</u>.

<u>Thm</u> X is connected iff ϕ, X are the only open sets which are both open and closed.

<u>Proof</u> $\Rightarrow$ suppose $A \neq \phi, X$ both open and closed then $X = A \cup A^c$ and $A \neq \phi$, $A^c \neq \phi$.

so A disconnected

$\Leftarrow$ suppose $X = A \cup B$ open nonempty, then A, A^c open and closed $\square$.

<u>Propn</u> If $A \subset X$, $B \subset X$ connected, and $A \cap B \neq \phi$, then $A \cup B$ is connected.

<u>Proof</u> suppose $A \cup B$ disconnected, then there is U, V disjoint open in X s.t. $A \cup B \subset U \cup V$ and $(A \cup B) \cap U \neq \phi$ and $(A \cup B) \cap V \neq \phi$. Suppose $A \cap U = \phi$ then ~~$U \cap B \subset B \setminus A$~~ $B \cap U \neq \phi$, and as $A \cap B \neq \phi$ $B \cap V \neq \phi$ so $B \subset U \cup V$ and $B \cap V \neq \phi$, $B \cap U \neq \phi \Rightarrow$ B is disconnected. ⨳.

<u>Corollary</u> product of connected sets is connected.

<u>Proof</u> $X \times Y$ $\boxplus$ $= \bigcup X \times \{y\} \cup Y \times \{x\}$.

Prop^n The continuous image of a connected set is connected.

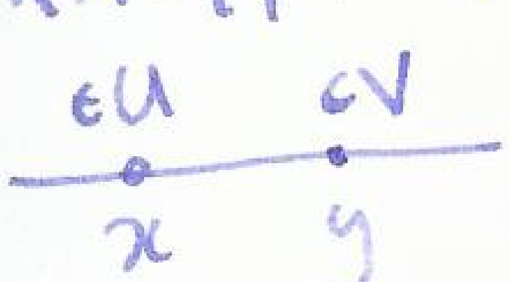
Proof $f: X \to Y$ cts. Spose $f(X)$ disconnected, then $f(X) \subset U \cup V$ open disjoint in Y with $U \cap f(X) \neq \phi$ and $V \cap f(X) \neq \phi \Rightarrow f^{-1}(U), f^{-1}(V)$ disjoint open non-empty in $X \Rightarrow X$ disconnected. $\square$.

Prop^n a subset $A \subset \mathbb{R}$ is connected iff A is an interval

i.e. $\begin{matrix}(a,b)\\ [a,b]\end{matrix}$ or $(-\infty, a\begin{matrix}]\\)\end{matrix}$ $\begin{matrix}(\\ [\end{matrix}a, \infty)$.

Proof $\Rightarrow$ spose A <u>not</u> an interval, then $\exists$ $p \in \mathbb{R} \setminus A$ and $a, b \in A$ with $a < p < b$. Then $A \subset (-\infty, p) \cup (p, \infty)$ disjoint open ~~such~~ sets with non-empty intersection with $A \Rightarrow A$ not connected.

$\Leftarrow$ spose A interval but not connected, then $A \subset U \cup V$ disjoint open sets with $A \cap U \neq \phi$ and $A \cap V \neq \phi$. Non-empty intersection $\Rightarrow$ there is $x \in U \cap A$ and $y \in A \cap V$, consider $[x, y]$



Let $p = \sup\{z \in [x,y] \mid z \in U\}$

suppose $p \in U$, then there is a base set $p \in (a,b) \subset U$ # $p = \sup\{z \in U \cap [x,y]\}$.

suppose $p \in V$ then there is a base set $p \in (a,b) \subset V$ # $p = \sup\{z \in U \cap [x,y]\}$ $\square$

Components

A connected component $C \subset X$ is a maximal connected subset of X.

Th^m The connected components of X form a partition of X (i.e. they are disjoint and their union is X).

Proof C_1, C_2 connected and $C_1 \cap C_2 \neq \phi \Rightarrow C_1 \cup C_2$ connected. $\square$

Locally connected

X is <u>locally connected</u> at $p \in X$ if every open set U containing p contains a connected open set containing p. (i.e. if the connected open sets containing p form a local base at p). X is locally connected if it is locally connected at every point.

Example: Hawaiian earring not locally connected.

§14 Complete metric spaces

X metric space. A sequence $\{a_n\}$ is a Cauchy sequence if for every $\epsilon > 0$ there is an N s.t. for all $n, m \geq N$ $d(a_n, a_m) \leq \epsilon$.

Prop$^{\underline{n}}$ Every convergent sequence in a metric space is a Cauchy sequence.

Def$^{\underline{n}}$ A metric space is complete if every Cauchy sequence converges.

Example $[0,1]$, $\mathbb{R}$ Non-example $(0,1)$

Equivalently: Th$^{\underline{m}}$ A metric space is complete iff every nested sequence of non-empty closed sets whose diameters tend to zero has a non-empty intersection.

Proof $\Rightarrow$ spse $A_1 \supset A_2 \supset \cdots$ closed s.t. $\lim d(A_n) \to 0$ for each A_n let $a_n \in A_n$

claim $\{a_n\}$ is a Cauchy sequence, so converges to p. claim $p \in \bigcap A_n$. spse not then $p \notin A_n$ for some n. A_n^c open so $p \in B(p,\epsilon) \subset A_n^c$ # $a_n \to p$.

$\Leftarrow$ let $\{a_n\}$ be a Cauchy sequence in X. Let $A_k = \{a_k, a_{k+1}, \dots\}$ claim $d(A_k) = d(\overline{A_k})$

so $\overline{A_n} \supset \overline{A_{n+1}}$ sequence of closed nested sets with $d(\overline{A_k}) \to 0$. so $\exists\, p \in \bigcap \overline{A_k}$

claim $a_k \to p$. for all $\epsilon > 0$ $\exists$ N st $d(a_n, p) \leq \epsilon$ for all $n \geq N$. $\square$.

Completions of metric spaces

Def$^{\underline{n}}$ A metric space $\overline{X}$ is a completion of X if $\overline{X}$ is complete, and X is isometric to a dense subset of X.

Example $\overline{(0,1)} = [0,1]$. $\overline{\mathbb{Q}} \cong \mathbb{R}$.

$\overline{\mathbb{Z}} = \mathbb{Z}$

Let $C[X]$ be the collection of all Cauchy sequences in X. define an equivalence relation $\{a_n\} \sim \{b_n\}$ iff $\lim\limits_{n\to\infty} d(a_n, b_n) = 0$ claim $\sim$ is an equivalence relation.

Let $\overline{X} = C[X]/\sim$ Define $d_{\overline{X}}(\{a_n\}, \{b_n\}) = \lim\limits_{n\to\infty} d(a_n, b_n)$ claim $d_{\overline{X}}$ well defined.

claim: $d_{\overline{X}}$ is a metric on $\overline{X}$.

claim: $i: X \hookrightarrow \overline{X}$ by $x \mapsto \{x\}$ constant sequence claim: i is an isometry.

claim: X is dense in $\overline{X}$. claim every Cauchy sequence converges in $\overline{X}$. so $\overline{X}$ completion of X.

$\underline{\text{Th}^m}$ A metric space X has a unique metric completion.

$\underline{\text{Proof}}$ Let Y be a completion of X, and $\bar{X}$ as above.

$X \hookrightarrow Y$ Let $y \in Y$, then there is a sequence $\{a_n\} \xrightarrow{\in X} y$ so define $f: Y \to \bar{X}$

$y \mapsto \{a_n\}$. check: well defined : spose $\underset{\in X}{b_n} \to y$, then $\lim_{n\to\infty} d_X(a_n, b_n) = 0$

$\Rightarrow \{a_n\} \sim \{b_n\}$ in $\bar{X}$.

check : f onto : let $x \in \bar{X}$ then there is a Cauchy sequence $a_n \to x$, but Y is complete so a_n converges in Y.

check: f isometry : let $x, y \in Y$ with $a_n \to x$ and $b_n \to y$

then $d_{\bar{X}}(f(x), f(y)) = d_{\bar{X}}(\{a_n\}, \{b_n\}) = \lim_{n\to\infty} d_X(a_n, b_n) = d_Y(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n) = d_Y(x,y)$ $\square$

(x dense in Y)

Completeness and compactness

$\underline{\text{Th}^m}$ A metric space X is compact iff complete and totally bounded.

$\underline{\text{Th}^m}$ Let X be a complete metric space $A \subset X$ is compact iff A is closed and totally bounded.

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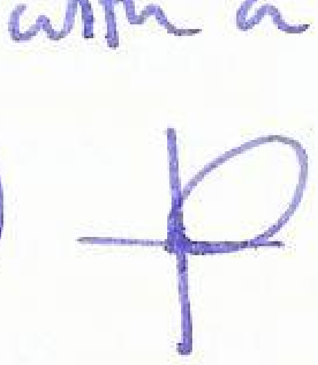
§1 Fundamental group

$\underline{\text{Overview}}$: (X, x_0) topological space with a basepoint. group elements: loops based at x_0

composition : ~~∞~~ $\underline{\text{Examples}}$ $(\mathbb{R}^2, (0,0))$  π_1 trivial.  $S^1 \vee S^1$. $\pi_1 = F_2$.

§1.1 Paths

$\underline{\text{Def}^n}$ A path is a cts $f: I \to X$ $I = [0,1]$ unit interval. A $\underline{\text{homotopy of paths relative to endpoints}}$ is a family of paths $f_t: I \to X$ s.t. $f_t(0) = x_0$ $f_t(1) = x_1$, const. independent of t, s.t. the corresponding map $F: I \times I \to X$ is cts.

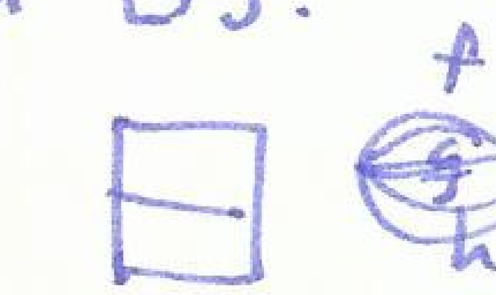
$(s,t) \mapsto f_t(s)$ 

We say f_0 and f_1 are $\underline{\text{homotopic}}$ notation $f_0 \simeq f_1$.

$\underline{\text{Example}}$ Any two paths f_0 and f_1 in $\mathbb{R}^n$ with same endpoints are homotopic via a

$\underline{\text{linear homotopy}}$ $f_t(s) = (1-t) f_0(s) + t f_1(s)$ } cts as mult. and addition in $\mathbb{R}^n$ are cts and composition of cts functions are cts.

$\underline{\text{Prop}^n}$ Homotopy of paths rel endpoints is an equivalence relation. notation $[f]$.

$\underline{\text{Proof}}$ reflexivity: $f \simeq f$ by constant homotopy $f_t(s) = f(s)$ 

symmetry : $f \simeq g$ by f_t then $g \simeq f$ by f_{1-t}.

transitivity: $f \simeq g$ and $g \simeq h$ by say f_t and g_t then define $h_t = \begin{cases} f_{2t} & 0 \le t \le \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}$

- cts: a function defined on the union of two closed sets is cts iff cts on each set.