Prop^n If $C(X,\mathbb{R})$ separates points, then X is Hausdorff.
↑ cts functions from X to $\mathbb{R}$.

Def^n X is completely regular iff. $F \subset X$ closed, $p \notin F$, then there is $f : X \to [0,1]$ cts. s.t. $f(p) = 0$, $f(F) = 1$.

Prop^n completely regular $\Rightarrow$ regular.

Th^m $C(X,\mathbb{R})$ separates points for a completely regular T_1 space.

§11 Compactness

An open cover of X is a collection U_i of open sets s.t. $X = \bigcup U_i$

Def^n $A \subset X$ is compact if every open cover of A has a finite subcover.

Th^m (Heine-Borel) A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Example
- $(0,1)$ is not compact $(\frac{1}{n}, 1-\frac{1}{n})$ open cover, no finite subcover.
- $\mathbb{R}$. $(n, n+2)$ open cover.

Th^m The continuous image of a compact set is compact.

Proof $f : X \to Y$ compact. Let U_i be an open cover of $f(X)$, then $f^{-1}(U_i)$ is an open cover of X, so has a finite subcover $f^{-1}(U_1), \dots f^{-1}(U_n)$, but then $U_1 \dots U_n$ is a finite open cover of $f(X)$ $\square$.

Th^m A closed subset of a compact space is compact.

Proof Let $A \subset X$ closed, compact. Let U_i be an open cover of A, the $U_i \cup A^c$ is an open cover of X, so has a finite subcover $U_1, \dots U_n, A^c$, so $U_1, \dots U_n$ is a finite cover of A. $\square$.

Compactness and Hausdorff spaces

Thm Every compact subset of a Hausdorff space is closed.

Proof $\underset{\text{compact}}{K} \subset X$ consider $p \in K^c$ and $q \in K$, so there are disjoint open sets U_q, V_q s.t. $p \in U_q$, $q \in V_q$, for all $q \in K$, so V_q form an open cover of K, so there is a finite subcover $V_1, \dots, V_n$, and $x \in \underset{\text{finitely many}}{\bigcap U_q} = W$ open disjoint from $V_1 \cup \dots \cup V_n \supset K$, so there is an open set $x \in W \subset K^c$ so K^c open $\Rightarrow K$ closed $\square$.

Thm A, B disjoint compact subsets of X Hausdorff. Then there are disjoint open sets U, V. $A \subset U$, $B \subset V$. $\square$. (exercise).

Corollary every compact Hausdorff space is normal.

Thm Let $f: X \to Y$ cts injective, X compact, Y Hausdorff, then X and $f(X)$ are homeomorphic.

Proof suffices to show $f(\text{closed})$ is closed. Let $K \subset X$ be closed, then K is compact, so $f(K)$ is compact $< f(X)$ compact and Hausdorff $\Rightarrow f(X)$ closed. $\square$.

Sequential compactness

Defn X is sequentially compact if every sequence contains a convergent subsequence.

Local compactness

Defn X is locally compact if every point in X has a compact neighbourhood.

Compactifications

Defn X is embedded in Y if X is homeomorphic to a subset of Y.

If Y is compact, then Y is a compactification of X.

Examples $\mathbb{R} \cup \{\pm\infty\}$; $\mathbb{R} \cup \{\infty\}$ describe topologies.

One point compactification

(X, T) topological space. (X_∞, T_∞) one point compactification

$X_\infty = X \cup \{\infty\}$ $\quad T_\infty = T \cup C$ where $C_K = K^c$ for all closed compact sets in X.

Claim X_∞ compact.

Thm If X is locally compact Hausdorff, then X_∞ is compact Hausdorff.

Proof spose $x, y \in X$ distinct, then Hausdorff $\Rightarrow$ open disjoint U, V with $x \in U$, $y \in V$

now spose $x \in X$, $\infty \in X_\infty$. Local compactness $\Rightarrow$ there is compact K with

$x \in K \subset X$ neighborhood $\Rightarrow \exists$ open U with $x \in U \subset \underset{\text{open compact}}{K} \subset X$. now U, K^c are

open sets separating x, ∞. $\square$

Compact metric spaces

Thm (X, d) metric space $A \subset X$. Then A is compact iff A is countably

compact $\Leftrightarrow$ A is sequentially compact.

Totally bounded sets

Defn $A \subset X$ metric space, $\epsilon > 0$. A finite set of points $\{x_1, \dots, x_n\}$ is an ϵ-net

for A if $A \subset \bigcup B(x_i, \epsilon)$.

Defn metric space $A \subset X$ is totally bounded if A has an ϵ-net for

every $\epsilon > 0$.

Example H = Hilbert space of ℓ_2-sequences. $d(a, b) = \sqrt{\sum (a_i - b_i)^2}$.

$A = \{e_i\}$ $\quad e_i = \langle 0, 0, \dots, 0, \underset{i}{1}, 0, \dots \rangle$. $\quad$ $\operatorname{diam}(A) = \sqrt{2}$.

no ϵ-net for $\epsilon = \frac{1}{2}$.

Propn Totally bounded $\Rightarrow$ bounded

Lemma sequential compactness $\Rightarrow$ totally bounded.

Proof spose not seq. compact, then $\exists$ seq $\{a_n\}$ with no convergent subsequence.

claim $\exists \epsilon > 0$ s.t. $B(a_i, \epsilon)$ all disjoint. Spose not, then for all $\epsilon > 0$ $\exists$ $B(a_i, 2\epsilon)$ with

$a_j \in B(a_i, 2\epsilon)$ Spose not totally bounded, then $\exists \epsilon > 0$ s.t. no finite set $\{a_i\}$

covers $A \Rightarrow \exists$ infinite set $\{a_i\}$ s.t. $B(a_i, \epsilon)$ all disjoint $\quad \{a_i\}$ has no convergent subseq $\square$