(X,T) is <u>second countable</u> if T has a countable basis B.

<u>Examples</u> • $(\mathbb{R}, \text{usual})$ choose $B = \{(a,b)\}$ $a,b \in \mathbb{Q}$.

∘ $(\mathbb{R}, \text{discrete})$ not second countable. • long line.

<u>Prop^n</u> 2nd countable $\Rightarrow$ 1st countable ☐

<u>Def^n</u> Let $A \subset X$ be a subset, and let $\mathcal{U}$ be a collection open sets.

$\mathcal{U}$ is an open cover for A if $A \subset \bigcup \mathcal{U}$.

If $\mathcal{U}$ contains a countable subset that covers A, then we say A has a <u>countable subcover</u>

<u>Th^m</u> (Lindelöf ①) $A \subset X$ 2nd countable. Then every open cover of A has a countable subcover.

<u>Th^m</u> (Lindelöf ②) X 2nd countable. Every basis contains a countable subbasis.

<u>Proof ①</u> X 2nd countable $\Rightarrow \exists B$ countable basis.

Let $\mathcal{U}$ be an open cover for A, i.e. for every $a \in A$ $\exists\ U_a \in \mathcal{U}$ with $a \in U_a$.

as B is a basis, $\exists\ B_p \in B$ with $p \in B_p \subset A_p$, so $A \subset \bigcup_p B_p$ countable

so for each B_p choose $B_p \subset A_p$, then $\{A_p\}$ is a countable subcover. ☐.

<u>Proof ②</u> Let B be a countable base for X. for each $B_n \in B$ there is a countable subcover $\{U_n\}$, by ①; but then the union of all these is countable and a basis for X ☐

<u>Separable spaces</u>

<u>Def^n</u> X is <u>separable</u> if X contains a countable dense subset.

<u>Example</u> $(\mathbb{R}, \text{usual})$ $\mathbb{Q}$ works.

<u>Nonexample</u> $(\mathbb{R}, \text{discrete})$ only dense set is $\mathbb{R}$ (not countable)

<u>Thm ①</u> (Lindelöf) $A \subset X$ 2nd countable. Every open cover of A has a countable subcover.

<u>Thm ②</u> (Lindelöf) X 2nd countable. Every base $\mathcal{B}$ contains a countable base.

<u>Proof ①</u> B_n countable base, $\{U_i\}_{i\in I}$ open cover of A. For each $a \in A$, there is B_n, U_i, s.t. $a \in B_{n(a)} \subset U_i$, so $\bigcup_{a\in A} B_{n(a)}$ is a countable cover. For each $B_{n(a)}$ pick $U_{n(a)}$ with $B_{n(a)} \subset U_n$ then U_n is a countable cover of A. □

<u>Proof ②</u> B_n countab base, $\mathcal{U}$ arbitrary base. For each $B_n \in \mathcal{B}$ there is a countable cover $\{U_n^{\cap B_n}\}$, countably many B_n, so union of all these covers gives a countable subset of $\mathcal{U}$, which is a base for X □.

<u>Separable spaces</u>

<u>Defn</u> X is <u>separable</u> if it has a countable dense subset.

<u>Example</u> $(\mathbb{R}, \text{usual})$ e.g. $\mathbb{Q}$.

<u>Nonexample</u> $(\mathbb{R}, \text{discrete})$ only dense set is $\mathbb{R}$.

<u>Propn</u> X 2nd countable $\underset{\not\Leftarrow}{\Rightarrow}$ X separable

<u>Proof</u> Let B_n be a countable base, and choose $b_n \in B_n$. <u>claim</u> $\{b_n\}$ is dense in X. Let $x \in X \setminus \{b_n\}$. For any open set $\mathcal{U}$ with $x \in \mathcal{U}$, there is a base set $x \in B_{n(x)} \subset \mathcal{U}$, so $B_{n(x)}$ contains $b_{n(x)} \neq x$, so x is an accumulation pt. □.

<u>Thm</u> X metric space, separable $\Rightarrow$ 2nd countable

<u>Proof</u> Let A be a countable dense set in X, let $\mathcal{B}$ be $B(a,q)$ $a \in A$, $q \in \mathbb{Q}$. (countable) <u>claim</u> $\mathcal{B}$ is a base. Let $x \in \mathcal{U}$ open, then there is $B(x,r) \subset \mathcal{U}$

A dense so choose a with $d(a,x) < \frac{r}{4}$

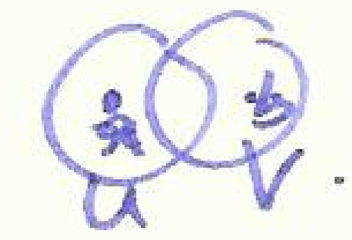
Then $x \in B(a,q) \subset B(x,r)$ for $\frac{r}{4} < q < \frac{r}{2}$ □

<u>Hereditary properties</u>: $A \subset X$ 2nd countable $\Rightarrow$ A 2nd countable

$A \subset X$ separable $\not\Rightarrow$ A separable.

§10 Separation properties

Def^n X is T_1 if for every pair of distinct points $a, b \in X$, there are open sets U, V with $a \in U$, $b \notin U$, $b \in V$, $a \notin V$.



Warning $U \cap V \neq \phi$ possible.

Example every metric space is T_1.

Non-example: $X = \{0, 1\}$ $\mathcal{T} = \{\phi, X, \{0\}\}$.

Th^m X is T_1 iff every point set $\{a\}$ is closed.

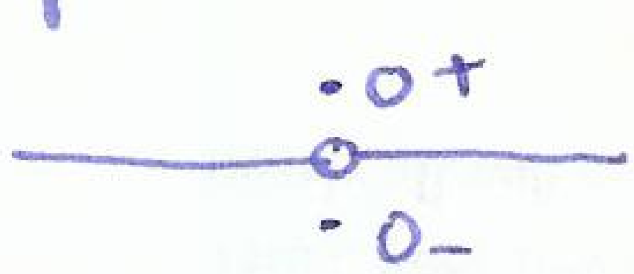
Proof $\Rightarrow$ show $\{p\}^c$ open. If $q \in \{p\}^c$ then $\exists$ open set U with $q \in U, p \notin U$
$\Rightarrow$ $q \in U \subset \{p\}^c \Rightarrow \{p\}^c$ open.

$\Leftarrow$ $\{a\}, \{b\}$ closed $\Rightarrow$ $\{a\}^c, \{b\}^c$ open. $\square$

Def^n X is T_2 or Hausdorff if for every distinct pair of points a, b, there is a disjoint pair of open sets U, V with $a \in U$, $b \in V$.

"every pair of points contained in disjoint open sets".

Non-example



open sets $(a,b) \not\ni 0$.
and $(a,b) \setminus 0 \cup 0+$
$(a,b) \setminus 0 \cup 0-$

or

$/\sim$
$(x,0) \sim (x,1)$ $x \neq 0$.
quotient topology.

Note $T_2 \Rightarrow T_1$

Th^m X Hausdorff, then every convergent sequence has a unique limit.

Th^m X 1st countable, then X Hausdorff $\Leftrightarrow$ every convergent sequence has a unique limit.

Proof suppose $a_n \to a$, $a_n \to b$. Let U, V be disjoint open sets containing a, b. As $a_n \to a$, $\exists$ N s.t. $a_n \in U$ for all $n \geq N$, but then $a_n \notin V \Rightarrow a_n \not\to b$. $\square$.

Def^n X is regular if you can separate closed sets and points by disjoint open sets.

i.e. $F \subset X$ closed, $p \notin F$, then there are disjoint open sets U, V with $F \subset U$, $p \in V$.

<u>Defn</u> X is T_3 iff X is T_1 and regular.

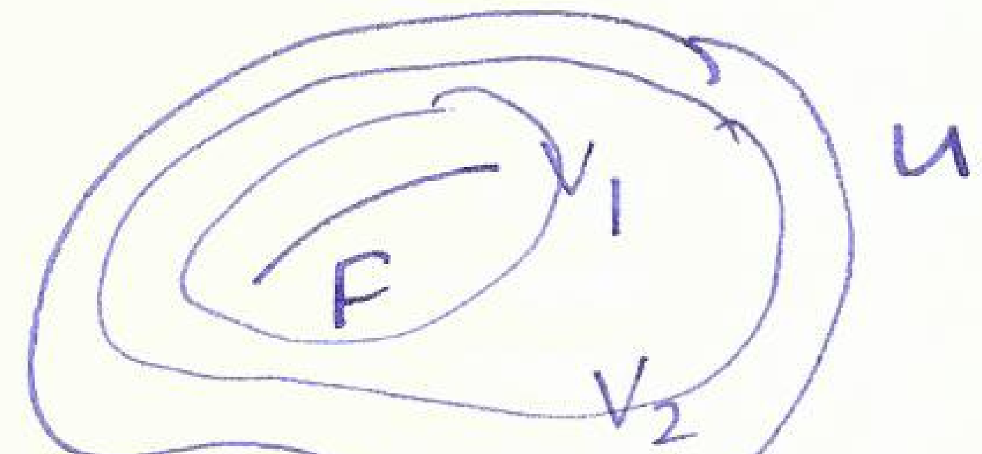
<u>Defn</u> X is <u>normal</u> if you can separate any two closed sets by disjoint open sets

i.e. $F_1, F_2 \subset X$ closed, $F_1 \cap F_2 = \phi$, then there are disjoint open sets U, V with $F_1 \subset U$, $F_2 \subset V$.

<u>Defn</u> X is T_4 iff X is T_1 and normal.

<u>Example</u> metric spaces are T_4. <u>Remark</u> $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$.

<u>Thm</u> X is normal iff: for every closed set F and every open set U, $F \subset U$, there is an open set V with $F \subset V \subset \overline{V} \subset U$.



<u>Proof</u> $\Rightarrow$ $\underset{\text{closed}}{F} \subset \underset{\text{open}}{U}$ $\quad U^c$ closed, and $F \cap U^c = \phi$

normal $\Rightarrow \exists$ disjoint open sets V_1, V_2 with $F \subset V_1$, $U^c \subset V_2$

claim: $F \subset V_1 \subset \overline{V}_1 \subset U$. $\quad$ <u>note</u> $V_1 \subset V_2^c$ closed so $\overline{V}_1 \subset V_2^c \subset U$.

$\Leftarrow$ F_1, F_2 disjoint closed sets, then $F_1 \subset F_2^c$ open. so there is U with $F_1 \subset U \subset \overline{U} \subset F_2^c$, so U and $\overline{U}^c$ disjoint open sets containing F_1 and F_2. $\square$.

<u>Thm</u> (Urysohn's Lemma) X normal, F_1, F_2 disjoint closed subsets, then there is a cts $f : X \to [0,1]$ s.t. $f(F_1) = 0$ and $f(F_2) = 1$.

<u>Proof</u> $F_1 \cap F_2 = \phi$ so $F_1 \subset F_2^c$ so by above, there is $U_{1/2}$ open with $F_1 \subset U_{1/2} \subset \overline{U}_{1/2} \subset F_2^c$ apply again: $F_1 \subset U_{1/4} \subset \overline{U}_{1/4} \subset U_{1/2} \subset \overline{U}_{1/2} \subset U_{3/4} \subset \overline{U}_{3/4} \subset F_2^c$

etc. let D = dyadic fractions $\frac{p}{2^n}$ lowest terms. in $[0,1]$. We get open sets $U_d, d \in D$ with property that if $d_1 < d_2$ $\quad \overline{U}_{d_1} \subset U_{d_2}$

define $f(x) = \begin{cases} \inf\{d \mid x \in U_d\} & x \notin F_2 \\ 1 & x \in F_2 \end{cases}$

claim: ~~ft~~ f is cts. want: f^{-1}(open) is open.

suffices to check for a subbase: $[0,a)$, $(b,1]$ subbase for $[0,1]$.

note $f^{-1}([0,a)) = \{\bigcup G_d \mid d<a\}$ open. $\square$.

Thm (Urysohn) Every 2nd countable $\underbrace{\text{normal } T_1}_{\equiv T_4}$ space is metrizable.

Proof (sketch) X 2nd countable T_4, homeomorphic to a subset of the Hilbert cube $I \subset \mathbb{R}^\infty$: $(a_1, a_2, \dots)$ s.t. $0 \leq a_n \leq \frac{1}{n}$

Hilbert space: $(a_1, a_2, \dots)$ $\sum a_n^2 < \infty$ l_2-metric: $d(p,q) = \sqrt{\sum_{n=1}^{\infty} |a_n - b_n|^2}$.

let B be a countable base $\{B_1, B_2, \dots\}$. assume $X \in B$.

normal $\Rightarrow$ for each B_i there is a B_j s.t. $\overline{B_j} \subset B_i$. consider all such countably many pairs:

P_n: $\overline{B_{j_n}} \subset B_{i_n}$. Urysohn's Lemma: there is $f_n : X \to [0,1]$ s.t.

$f_n(\overline{B_{j_n}}) = 0$ and $f_n(B_{i_n}^c) = 1$.

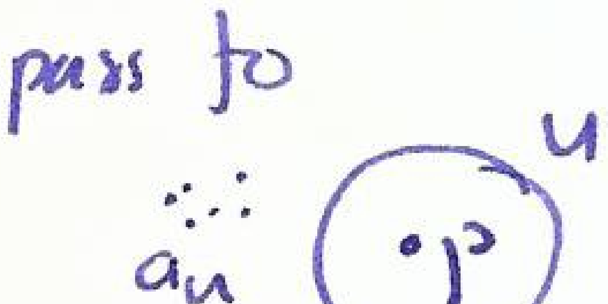
now define $f : X \to I$ by $f(x) = \left(\frac{f_1(x)}{2}, \frac{f_2(x)}{2^2}, \frac{f_3(x)}{2^3}, \dots \right)$.

claim f injective: the f_n separate points.  so some pair P_n. corresponding f_n

has $f_n(x) = 0$, $f_n(y) = 1$.

claim f cts at $p \in X$: given $\epsilon > 0$ there is open set U with $\|f(p) - f(q)\|^2 < \epsilon^2$ for all $q \in U$. — only need to worry about first $n \sim \log_2(1/\epsilon)$ functions - each cts so $\exists$ open sets $U_1, \dots, U_n$ with this property

claim f^{-1} cts: show if $a_n \not\to p$ then $f(a_n) \not\to f(p)$.

$a_n \not\to p$ so there is an open set $p \in U$ containing finitely many a_n, ~~to~~ pass to subsequence, U contains no term of a_n. so there is a fixed pair 

P_m $p \in \overline{B_{j_m}} \subset B_{i_m}$ with $f_m(p) = 0$ $f_m(a_n) = 1$, so $\|f(a_n) - f(p)\|^2 \geq \frac{1}{2^m}$ $\forall n$ $\square$

Functions that separate points

Let $\{F_i\}$ F be a collection of cts function $f : X \to Y$ F separates points if for every $a, b \in X$ $a \neq b$, there is $f \in F$ s.t. $f(a) \neq f(b)$.