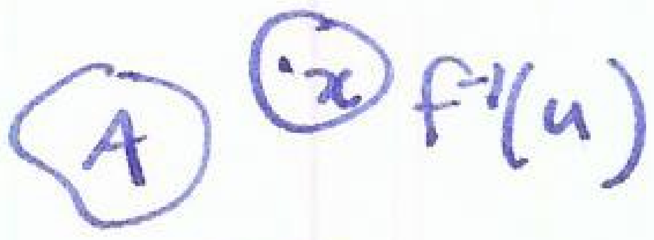
Th^m $f: X \to Y$ is cts iff $f(\overline{A}) \subset \overline{f(A)}$. (*)

Proof $\Rightarrow$ let $x \in X$ with $f(x) \notin \overline{f(A)}$, then $\exists$ open set U s.t. $f(x) \in U \subset \overline{f(A)}^c$

$f(x) \in U \subset \overline{f(A)}^c$

$$\underset{\text{open}}{x \in f^{-1}(U)} \subset f^{-1}\left(\overline{f(A)}^c\right) \subset f^{-1}(f(A)^c) \subset A^c$$



$\Rightarrow x \notin \overline{A}$.

$\Leftarrow$ let $B \subset Y$ closed then $\cancel{f(\overline{f^{-1}(B)}) \subset}$ ~~[illegible]~~

use (*) with $A = f^{-1}(B)$: $f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} = \overline{B} = B$

$\Rightarrow \overline{f^{-1}(B)} = f^{-1}(B) \Rightarrow f^{-1}(B)$ closed $\Rightarrow f$ cts. $\square$

Continuity at a point p point $\in X$. N_p open neighbourhoods of p.

$f: X \to Y$ is continuous at $p \in X$ if for every $U \in N_{f(p)}$ $f^{-1}(U) \in N_p$. **Th^m** $f: X \to Y$ is cts iff cts at every point $p \in X$.

Sequential continuity

$f: X \to Y$ is sequentially continuous at $p \in X$ iff $a_n \to p$ $\Rightarrow f(a_n) \to f(p)$

Prop^n $f: X \to Y$ cts at $p \in X \Rightarrow f$ sequentially cts at $p \in X$.

Converse doesn't hold: $(\mathbb{R}, T)$ T = sets with countable complement and ϕ.

Note $a_n \to p \in \mathbb{R}$ iff $a_n = \{a_1, a_2, \dots, p, p, \dots\}$.

proof suppose not, then there is a sequence $a_n \to p$ with no $a_n = p$. but then $U = \mathbb{R} \setminus \bigcup a_n$ is open, and $a_n \to p$ means for any open set U $p \in U$ $\exists N$ s.t. for all $n \geq N$ $a_n \in U$ #.

so every function $f: (\mathbb{R}, T) \to (X, T_X)$ is sequentially continuous.
but e.g. $id: (\mathbb{R}, T) \to (\mathbb{R}, \text{standard})$ not cts. $(f^{-1}((0,1))$ not open in $T)$.

Open and closed functions

Def^n $f: X \to Y$ is <u>open</u> if the image of every open set is open

A open $\Rightarrow f(A)$ open.

$f: X \to Y$ is <u>closed</u> if the image of every closed set is closed.

A closed $\Rightarrow f(A)$ closed.

Homeomorphisms

Def^n $f: X \to Y$ is a <u>homeomorphism</u> if f is a <u>bijection</u> and <u>both</u> f and f^{-1} are <u>continuous</u>.

Remark The relation of homeomorphism gives an equivalence relation on the class of topological spaces.

Examples · $(0,1)$ homeomorphic to $\mathbb{R}$ homeo to $(0,\infty)$

· S^1 O homeomorphic to square □

Topological properties

A property is <u>topological</u> or a <u>topological invariant</u> if whenever (X,T) has the property any homeomorphic space has the property.

Non-examples: Length, boundedness, Cauchy sequence convergence (completeness)

Examples: connectedness.

Def^n (X,T) is <u>disconnected</u> if X is the union of two disjoint non-empty sets.

Prop^n disconnectedness is a topological property.

Proof $f: X \to Y$ homeo and $Y = A \cup B$ open nonempty, disjoint then $f^{-1}(A), f^{-1}(B)$ open non-empty, disjoint and $X = f^{-1}(A) \cup f^{-1}(B)$ so X is disconnected.

Quotient topology

(X,T) space $\sim$ equivalence relation on X. Let $q: X \to X/\sim$

quotient topology on $X/\sim$ is: $A \subset X/\sim$ is open iff $q^{-1}(A)$ open in X.

Example $I = [0,1]$. $\sim$: $0 \sim 1$ $\quad I \to S^1$.

$I^2 = [0,1]\times[0,1]$ $\sim$: $(0,x)\sim(1,x)$, $(x,1)\sim(x,0)$ gives $S^1 \times S^1 = T^2$.

Countability

X is first countable if for every point $p \in X$ there is a countable local basis $\mathcal{B}_p$ at p.

Thm $f: X \to Y$, X first countable, is continuous iff it is sequentially continuous.

Propⁿ If $\mathcal{B}_p$ is a countable basis at $p \in X$, then there is a nested countable basis ~~$\mathcal{B}_p$~~ $U_1 \supset U_2 \supset U_3 \supset \dots$

Proof ~~set~~ number $\mathcal{B}_p = \{B_1, B_2, B_3, \dots\}$. set $U_1 = B_1$, $U_2 = B_1 \cap B_2$, $U_3 = B_1 \cap B_2 \cap B_3$ etc.

so $U_1 \supset U_2 \supset U_3 \supset \dots$ open!

and $p \in U_i$ for each i

check local bases: let $p \in V$ open, then $\exists\, n$ s.t. $p \in B_n \subset V$

but then $p \in U_n \subset B_n \subset V$ so $\mathcal{U}_p$ is local nested basis. $\square$.

Propⁿ Let $\mathcal{B}_p$ be a nested local basis and $a_n \in B_n$. then $a_n \to p$.

Proof for any open set $p \in \underset{\text{open}}{U}$ $\exists\, n$ s.t. $p \in B_n \subset U$ so as $B_n \supset B_{n+1} \supset B_{n+2} \supset \dots$

$a_i \in B_n \subset U$ for all $i \geq n$ so $a_n \to p$. $\square$.

Proof (Thm) cts $\Rightarrow$ seq cts. (already done).

$f: X \to Y$ spose $\underset{\text{open}}{A \subset Y}$ but $f^{-1}(A)$ not open, then $\exists\, p \in f^{-1}(A)$ s.t. for any open set $p \in U$ $\exists\, x \in U \setminus f^{-1}(A)$. Choose $U = B_n$ for some ordered local bases nested, and let $a_n \in B_n \setminus f^{-1}(A)$ then $a_n \to p$ by Propⁿ. but then $f(a_n) \notin A$ for all n so $f(a_n) \not\to f(p)$ # $\square$.

Example ($\mathbb{R}$, cofinite topology) ($\mathbb{R}$, cocountable topology) not first countable.