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Zariski topology
Gromov-Hausdorff topology

Examples ($\mathbb{R}$, standard topology)

($\mathbb{R}$, trivial topology) $T = \phi, \mathbb{R}$. (indiscrete topology)

($\mathbb{R}$, discrete topology) $T =$ all subsets.

(X, cofinite topology) $T =$ sets whose complements are finite, also ϕ.

(X, T_1) (X, T_2) topologies then $(X, T_1 \cap T_2)$ is a topology.

Remark in general $T_1 \cup T_2$ is not a topology Example $X = \{1, 2, 3\}$.

$T_1 = \phi, \{1\}, X$

$T_2 = \phi, \{2\}, X$.

Observation if every $x \in A$ contains an open set $x \in U \subset A$ then A is open.

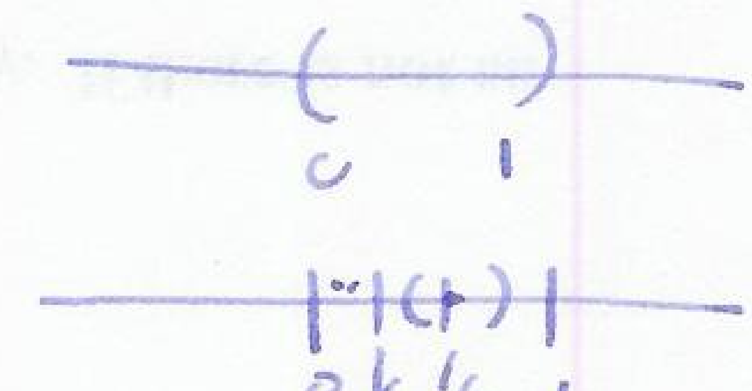
Accumulation points

X topological space $A \subset X$ a point $p \in X$ is an accumulation point of A if every open set containing p contains a point of A different from p.

i.e. $p \in U$ open then $(U \setminus p) \cap A \neq \phi$.

Example $A = (0,1) \subset \mathbb{R}$ has accumulation points: $[0,1]$

$A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ has accumulation points: $\{0\}$.



$\mathbb{Z} \subset \mathbb{R}$ has no accumulation points.

Thm (Bolzano-Weierstrass) If $A \subset \mathbb{R}$ is any bounded infinite set of real numbers then A has at least one accumulation point.

Closed sets

X topological space $A \subset X$ is closed iff $X \setminus A = A^c =$ complement of A is open

Examples $[0,1] \subset \mathbb{R}$ $\{0\} \subset \mathbb{R}$ any finite collection of points in $\mathbb{R}$. $(0,1]$ not open or closed.

ϕ, X both open and closed (clopen)

(X, discrete topology) every set both open and closed.

Thm X topological space, then the closed sets satisfy:

1) ϕ, X closed

2) arbitrary intersection of closed sets is closed

3) finite unions of closed sets are closed.

Thm $A \subset X$ is closed iff A contains its set of accumulation points.

Proof $\Rightarrow$ A closed $\Rightarrow$ A^c open. $p \in A^c$ open $\Rightarrow \exists$ open set $p \in A^c$ with $A^c \cap A = \phi$

so p is not an accumulation point. so $acc(A) \subset A$.

$\Leftarrow$ suppose $acc(A) \subset A$ claim A^c is open. proof: let $p \in A^c$ then $p \notin acc(A)$ so $\exists$

open set $p \in U$ with $U \cap A = \phi$ but then $p \in U \subset A^c \Rightarrow A^c$ open $\Rightarrow A$ closed.

Closure of a set

$A \subset X$ the closure of A, written $\bar{A}$ is the intersection of all closed sets containing A.

Prop 1) $\bar{A}$ is closed.

2) A is closed iff $A = \bar{A}$.

Proof 1): arbitrary intersection of closed sets is closed $\square$

2) $\Leftarrow$: from 1). $\Rightarrow$ $A \in$ all closed sets containing A, so $\bigcap$ gives A $\square$

Thm $\bar{A} = A \cup acc(A)$

Proof: spose $x \notin A \cup acc(A)$, then $\exists$ open set $x \in U$ s.t. $U \cap A = \phi$, so $x \notin U^c$ closed,

contains A, so $x \notin \bar{A}$. (this shows $\bar{A} \subset A \cup acc(A)$)

Examples $\overline{(0,1)} \subset \mathbb{R} = [0,1]$ $\quad \overline{B(x,r)} = \{y \mid d(x,y) \leq r\}$. $\quad \bar{\mathbb{Q}} \subset \mathbb{R} = \mathbb{R}$.

Defn $A \subset X$ is dense in $B \subset X$ if $B \subset \bar{A}$.

$x \in A$ lies in the interior A° of A if there is an open set U s.t. $x \in U \subset A$.

Propn A° is open; A° is the largest open subset of A; A is open iff $A^\circ = A$.

Defn the exterior of A, $ext(A)$ is $int(A^c)$

the boundary (or frontier) of A is $bound(A)$ $b(A)$ = "everything else" i.e. $b(A) = X \setminus (A^\circ \cup ext(A))$

Thm $\bar{A} = A^\circ \cup b(A)$.

Proof: recall $\bar{A} = A \cup acc(A)$. If $x \in A$ then $x \notin ext(A) \Rightarrow A \subset A^\circ \cup b(A)$

If $x \in acc(A)$ then $x \notin ext(A) \Rightarrow \bar{A} \subset A^\circ \cup b(A)$.

If $x \in b(A) \setminus A \Rightarrow$ every open set containing x contains an element of $A \Rightarrow A^\circ \cup b(A) \subset \bar{A}$ $\square$

Example $B(x,r)$: $b(B(x,r)) = \{y \mid d(x,y) = r\}$.

$\mathbb{Q} \subset \mathbb{R}$ $\quad int(\mathbb{Q}) = \mathbb{Q}^\circ = \phi \quad ext(\mathbb{Q}) = \phi. \quad b(\mathbb{Q}) = \mathbb{R}$.

Def^n $A \subset X$ is nowhere dense if $\operatorname{int}(\bar{A}) = \emptyset$.

Examples $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$, Cantor set.

Def^n A neighbourhood N for $p \in X$ is a set N s.t. there is an open set U with $p \in U \subset N$.

The class of all neighbourhoods N_p is called the neighbourhood systems of p.

Remark: neighbourhoods need not be open (or closed!).

Neighbourhood properties (axioms) 1) $N_p \neq \emptyset$ and for each $N \in N_p$, $p \in N$

2) If $A, B \in N_p$ then $A \cap B \in N_p$.

3) If $A \in N_p$ and $A \subset B$, then $B \in N_p$

4) each $A \in N_p$ contains a neighbourhood $B \in N_p$ s.t. B is a neighbourhood for each point $x \in B$.

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1) N_p is not empty, and every neighbourhood contains p.

2) the intersection of any two elements of N_p lies in N_p

3) every set containing an element of N_p lies in N_p

4) each $A \in N_p$ contains a neighbourhood $B \in N_p$ s.t. B is a neighbourhood for each $x \in B$.

Convergent sequences

A sequence $\{a_1, a_2, \dots\}$ converges to a point $b \in X$: write $\lim_{n \to \infty} a_n = b$ $a_n \to b$ if for every open set U with $b \in U$, there is a number N s.t. $a_n \in U$ for all $n \geq N$.

Remark: in a metric space makes sense to talk about a Cauchy sequence, i.e. $\{a_n\}$ s.t. for every $\epsilon > 0$ there is an N s.t. $|a_n - a_m| \leq \epsilon$ for all $m, n \geq N$, and we say a metric space is complete if every Cauchy sequence converges].

Comparing topologies

Let T_1 and T_2 be topologies on X space $T_1 \subset T_2$ (i.e. every T_1-open set is open in T_2) then we say T_1 is smaller, coarser, weaker than T_2

T_2 is larger, finer, stronger than T_1

Note: collection of all topologies is partially ordered by inclusion.

Example trivial $\subset$ cofinite $\subset$ discrete.

Subspaces, relative topologies

(X,T) topological space $A \subset X$, then the relative topology on A is

$T_A = \{ U \cap A \mid U \in T \}$. Exercise : check $\{A, T_A\}$ is a topology on A.

Example $[0,1] \subset \mathbb{R}$. then $[0,\frac{1}{2})$ is open in relative topology on $[0,1]$.

Equivalent def^{ns} of topologies

- X non empty, for each $x \in X$ there are neighbourhood subsets N_x satisfying nbd axioms.
- Kuratowski closure axioms.

Bases and subbases

Let (X,T) be a topological space. A collection of open sets $B \subset T$ is a base for T if 1) every open set is a union of elements of B.

2) for any point $p \in U$ open, there is a set $V \in B$ s/t $p \in V \subset U$.

Examples · $\mathbb{R}$, usual topology: $B =$ open intervals is a basis.

· $\mathbb{R}^2$, usual topology: $B =$ open $I \times J$, I, J open intervals.

· (X,d) metric space topology $B = \{ B(x,r) \mid x \in X \; r > 0 \}$.

Subbases (X,T) topological space $S \subset T$ is a subbases if finite intersections of elements of S form a bases.

Remark let $A \subset \mathcal{P}(X)$ (arbitrary) A generates a topology on X: let B be the bases formed by finite intersections of elements of A, then B generates a topology T.

A local base at $p \in X$ is a collection B_p of open sets containing p s.t. for every open set U with $p \in U$, there is an open set $B \in B_p$ with $B \subset U$.

Remark many different metrics give same topology

Continuity

(X, T_X) (Y, T_Y) topological spaces

Def^{n} $f : X \to Y$ is continuous if $U \subset Y$ open implies $f^{-1}(U)$ open in X.

Prop^{n} $f : X \to Y$ is continuous if $U \in B$ bases (or subbases) then $f^{-1}(U)$ open in X.