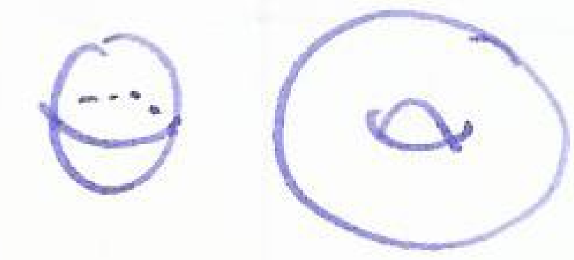
Topology I Math 70700

Joseph Maher joseph.maher@csi.cuny.edu

http://www.math.csi.cuny.edu/~maher

Office hours
TTh 11:30 – 12:30
4214 – 08

Motivation

Q: when are two spaces the same? e.g. 

find (algebraic) invariants – if they're different the two spaces are different.

e.g # connected components. S^0 : • • 2 components
S^1 :  1 components.

I interval  $I \setminus pt$ 2 component
S^1 O $S^1 \setminus pt$ 1 component.

higher dimensional connectedness:  disc : simply connected (1-connected)
every map $S^1 \to X$ can be shrunk to a pt

 S^1 : not simply connected.

two versions : homotopy groups (non-abelian)
homology groups (abelian) ← computable, hence useful..

Plan: 1. Review of general topology (Schaum).
2. Fundamental group. (Hatcher §0,1).
3. Homology (Hatcher §2).

Sets. $\{1, 2, \ldots\}$.
$\{\text{positive integers}\} = \mathbb{N}$
$\{\text{real numbers}\} = \mathbb{R}$
ϕ or $\{\}$ empty set.

$P(A)$ power set = all subsets
$P(\{0,1\}) = \phi, \{0\}, \{1\}, \{0,1\}$.

operations. $\cap$ intersection
$\cup$ union.
$\in$ member of. e.g. $1 \in \{0,1\}$.
$\notin$ not member of $2 \notin \{0,1\}$.
$\subset$ subset.
$\setminus$ complement.
$\times$ products.

Q: why does set theory exist?

A: Hilbert's paradox: the collection of all sets is not a set.

we say a set is inclusive if it contains itself. (e.g. the set of all sets is inclusive; $\mathbb{N}$ is not inclusive.).

Consider the set X of all non-inclusive sets. Q is X inclusive?

~~X~~ X inclusive: $X \in X$ but then $X \notin X$ as X is non-inclusive sets #.
X non-inclusive: $X \notin X \Rightarrow X \in X$ (non-inclusive sets) #.

Solution: the collection of all sets is called a class.
groups
spaces.

Cardinality

Two sets A and B have the same cardinality if there is a bijection $f: A \to B$.

Examples $\{0,1\} \leftrightarrow \{3,4\}$
$\mathbb{N} \leftrightarrow$ ~~Z~~ $\{\text{even integers}\}$ $\{1, 2, 3, 4, \ldots\}$.
$\{2, 4, 6, 8, \ldots\}$.

~~Exercise~~ A set is countable if it is finite, or has the same cardinality as $\mathbb{N}$

Exercise: show $\mathbb{Z}$, $\mathbb{Q}$, $\{\text{algebraic numbers}\}$ are countable.

Uncountable sets: $\mathbb{R}$, $[0,1]$ are uncountable.

Cantor's diagonal argument: write elements of $[0,1]$ as (unique) decimal numbers. (i.e. always write $0.49999\ldots$ instead of $0.5000\ldots$)

spose $[0,1]$ is countable : $1 \leftrightarrow 0.\boxed{4}321\cdots$
(i.e. bijection with $\mathbb{N}$) $2 \leftrightarrow 0.9\boxed{1}64\cdots$
$3 \leftrightarrow 0.00\boxed{1}2\cdots$

construct a number x whose nth digit is the nth digit of the nth number $+1$ (mod 10). Then x does not appear on the list. ↯.

<u>Exercise</u> show $\mathbb{R}$, $[0,1]$, $\mathbb{R}^2$, $\mathbb{R}^n$ have the same cardinality.

<u>Thm</u> (Cantor) $P(A)$ has cardinality greater than A.

<u>Continuum hypothesis</u> : $P(\mathbb{N}) = \mathbb{R}$. (in cardinality). (independent of standard axioms of set theory)

<u>Axiom of choice</u> : the product of a collection of non-empty sets is non-empty set

equivalent to : <u>well ordering principle</u> : every set can be well-ordered.

<u>Zorn's lemma</u> : X non-empty partially ordered set in which every totally ordered set has an upper bound. Then X contains at least one maximal element.

<u>Partial order</u> on a set X : is a relation $\leq$ on elements of X s.t.

1) $a \leq a$.
2) $a \leq b$ and $b \leq a \Rightarrow a = b$
3) $a \leq b$ and $b \leq c \Rightarrow a \leq c$

<u>Example partial</u>

<u>Total order</u> as above but for every $a, b \in X$ either $a \leq b$ or $b \leq a$.

<u>Example</u> totally ordered : $\mathbb{N}$, $\mathbb{R}$. <u>Note</u> : a subset $A \subset X$ inherits an order from X.

An element $a \in X$ is <u>maximal</u> if $a \leq x \Rightarrow x = a$
<u>minimal</u> $x \leq a \Rightarrow x = a$

$A \subset X$ has an <u>upper bound</u> $m \in X$ if $a \leq m$ for all $a \in A$.
" <u>lower bound</u> $l \in X$ " $l \leq a$ " $a \in A$

Lines, planes, topologies

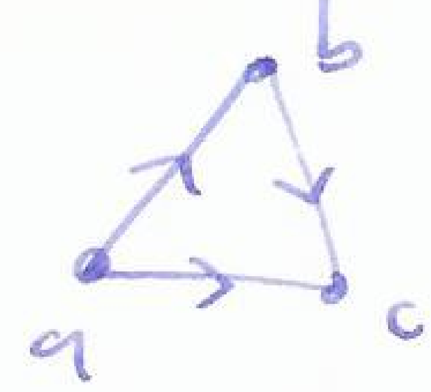
Motivation: $f: \mathbb{R} \to \mathbb{R}$ continuous.

- if for all $\epsilon > 0$ there is $\delta > 0$ s.t. if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$

This definition works in any metric space:

Defn (X,d) is a metric space if $d: X \times X \to \mathbb{R}$

- $d(a,b) \geq 0$ and $d(a,a) = 0$
- $d(a,b) = d(b,a)$ (symmetry)
- $d(a,b) + d(b,c) \geq d(a,c)$ (triangle inequality)
- $d(a,b) = 0 \Rightarrow a = b$



Examples
- $(\mathbb{R}, |\cdot|)$ $d(a,b) = |a-b|$.
- $(\mathbb{R}^2, |\cdot|)$ $d((a_1,a_2),(b_1,b_2)) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}$.
  $d((a_1,a_2),(b_1,b_2)) = \max\{|a_1-b_1|, |a_2-b_2|\}$.
- $(X, \text{trivial metric})$ $d(a,b) = \begin{cases} 0 & \text{if } a=b \\ 1 & \text{otherwise} \end{cases}$
- any subset $Y \subset X$ with $d|_Y$ as the distance.
- $C([0,1])$ continuous functions on $[0,1]$ $d(f,g) = \int_0^1 |f(x) - g(x)|\,dx$.

  or $d(f,g) = \sup |f(x) - g(x)|$

Better defn of cts function. $f: X \to Y$ cts if for every open set $A \subset Y$ $f^{-1}(A)$ is open in X.

Open sets (metric space version)

we shall write $B(x,r)$ for the open ball of radius r about x.
i.e. $B(x,r) = \{y \in X \mid d(x,y) < r\}$. in $\mathbb{R}$: these sets are just open intervals (a,b).

Defn A set $U \subset X$ is open if for every $x \in U$ there is an r s.t. $B(x,r) \subset U$.

Example • $B(x,r)$ is open (itself).

check: 

let $y \in B(x,r)$, choose $B(y,s)$ with $s = \cancel{d} \frac{r - d(x,y)}{2}$

show $B(y,s) \subset B(x,r)$: spose $z \in B(y,s)$ then

$$d(z,x) \leq d(z,y) + d(y,x) \quad \text{(triangle inequality)}$$
$$\leq \frac{r - d(x,y)}{2} + d(x,y) = \frac{r + d(x,y)}{2} < r$$

• ϕ empty set is open. (so works for $r = 0$! $B(x,0) = \phi$).

• in $\mathbb{R}$: (a,b) is open, as are $(0,\infty)$, $\mathbb{R}$.

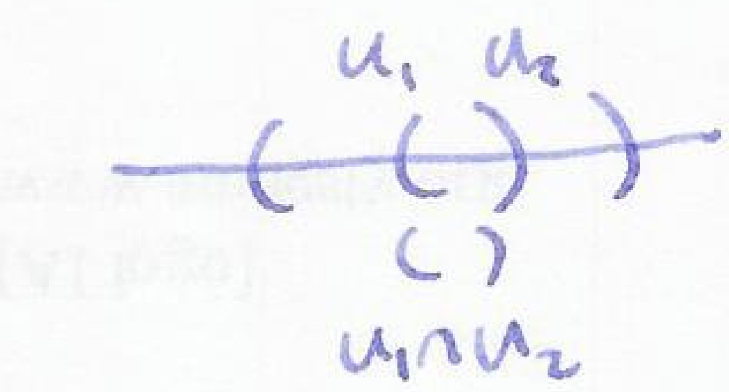
not open: $[0,1]$ $(0,1]$ $[0,\infty)$, $\{0\}$.

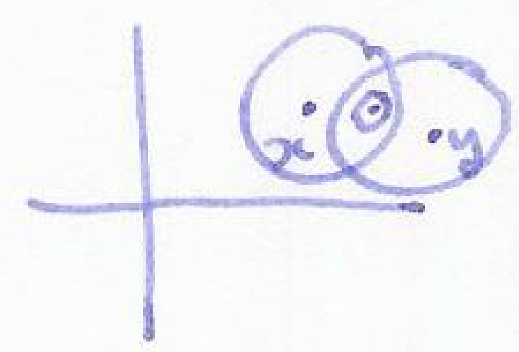
key properties:

Thm 1) The union of any collection of open sets is open.
2) The intersection of any finite number of open sets is open.

Example consider $\bigcap_n \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ not open. (finite condition necessary)

Proof 1) $\{U_i\}_{i\in I}$ open sets. $U = \bigcup_{i \in I} U_i$. if $x \in U$ then $x \in U_i$ for some i.
so $\exists\ B(x,r) \subset U_i \subset U$ ☑

2) **Examples** 



$B(x,r) \cap B(y,s)$ not a ball in general.

Proof: consider $x \in U_1 \cap U_2$ open sets. then $\exists\ r_i$ s.t. $x \in B(x,r_1) \subset U_1$
$x \in B(x,r_2) \subset U_2$

choose $r = \min\{r_1, r_2\}$ then
$x \in B(x,r) \subset U_1$
$x \in B(x,r) \subset U_2$ so $x \in B(x,r) \subset U_1 \cap U_2$ □

Exercise Show f^{-1}(open) open equivalent to standard $\epsilon\delta$ defn of cts.

General case X (non-empty) set. A topology on X is a collection of subsets T

such that:
1) $\phi, X \in T$
2) arbitrary union of elements of T lies in T
3) finite intersection of elements of T lie in T.