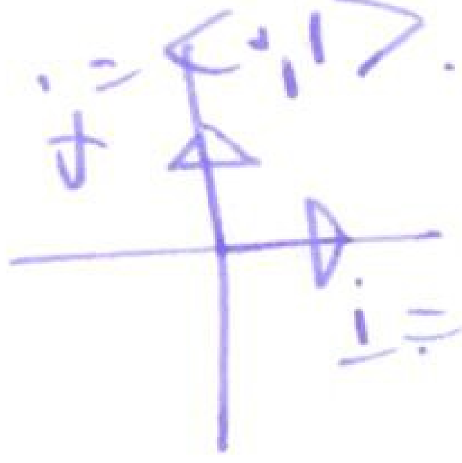
**Useful properties**

$$\underline{v}+\underline{w} = \underline{w}+\underline{v}$$
$$\underline{u}+(\underline{v}+\underline{w}) = (\underline{u}+\underline{v})+\underline{w}$$
$$\lambda(\underline{v}+\underline{w}) = \lambda\underline{v}+\lambda\underline{w}.$$

**Special vectors**




A linear combination of $\underline{v}, \underline{w}$ is a vector of the form $c_1\underline{v}+c_2\underline{w}$.

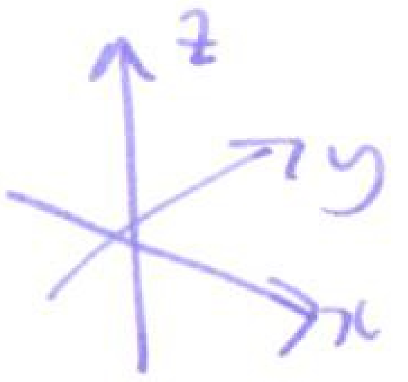
every vector $\underline{v} \in \mathbb{R}^2$ is a linear combination of $\underline{i}, \underline{j}$.

**unit vectors** $\underline{v}$ is a unit vector if $\|\underline{v}\| = 1$

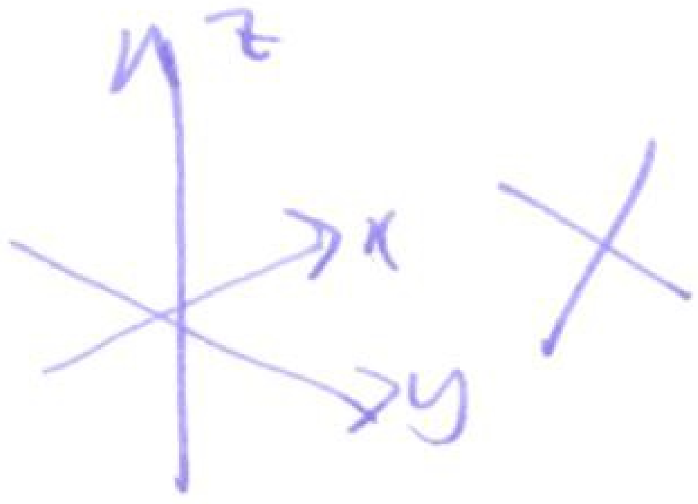
for any vector $\underline{w}$ $\quad \underline{w} \cdot \frac{1}{\|\underline{w}\|}$ is a unit vector.

**triangle inequality** $\|\underline{v}+\underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\|$.

## §12.2 Vectors in $\mathbb{R}^3$




right hand rule ✓.




wrong

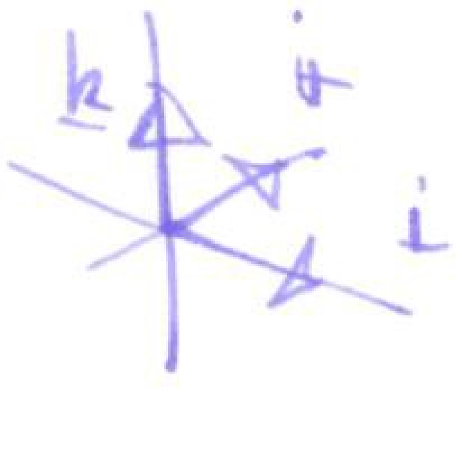
vectors have length and direction.

$\underline{v} = \langle x, y, z\rangle. \qquad \|\underline{v}\| = \sqrt{x^2+y^2+z^2}$.

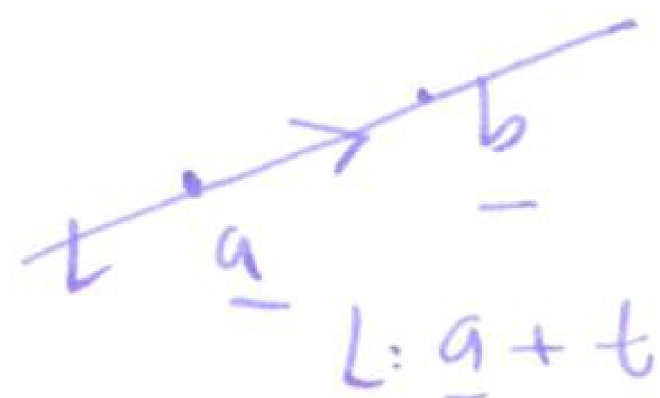
scalar multiplication: $\lambda\underline{v} = \lambda\langle a, b, c\rangle = \langle \lambda a, \lambda b, \lambda c\rangle$

**addition** $\underline{v}+\underline{w} = \langle a_1, b_1, c_1\rangle + \langle a_2, b_2, c_2\rangle = \langle a_1+a_2, b_1+b_2, c_1+c_2\rangle$.

**standard vectors**




$\underline{i} = \langle 1,0,0\rangle$
$\underline{j} = \langle 0,1,0\rangle$
$\underline{k} = \langle 0,0,1\rangle$




$L: \underline{a}+t(\underline{b}-\underline{a})$.

**Parametric equation of a line**


$L: \underline{a}+t\underline{v}$

## §12.3 Dot products

**Def$^{\underline{n}}$** $\quad \underline{v}\cdot\underline{w} = \langle a_1, b_1, c_1\rangle \cdot \langle a_2, b_2, c_2\rangle = a_1a_2 + b_1b_2 + c_1c_2$

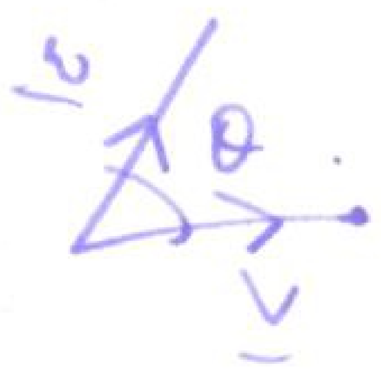
**Remark** $\quad \underline{v}\cdot\underline{w} = \underline{w}\cdot\underline{v}$.

**Useful properties**: $\quad \underline{0}\cdot\underline{v} = \underline{v}\cdot\underline{0} = 0$

$$(\lambda\underline{v})\cdot\underline{w} = \underline{v}\cdot(\lambda\underline{w}) = \lambda(\underline{v}\cdot\underline{w}).$$

$$\underline{u}\cdot(\underline{v}+\underline{w}) = \underline{u}\cdot\underline{v} + \underline{u}\cdot\underline{w}$$

$$\underline{v}\cdot\underline{v} = \|\underline{v}\|^2.$$




**Th$^{\underline{m}}$** $\quad \underline{v}\cdot\underline{w} = \|\underline{v}\|\,\|\underline{w}\|\cos\theta \iff \cos\theta = \dfrac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\,\|\underline{w}\|}$

**Perpendicular vectors** $\quad \underline{v}\perp\underline{w}$ iff $\theta = \frac{\pi}{2}$ iff $\underline{v}\cdot\underline{w} = 0$

**Projections**.


can write $\underline{w}$ as $\underline{w} = \underbrace{\underline{a}}_{\substack{\text{parallel to } \underline{v} \\ \text{i.e. } \lambda\underline{v}.}} + \underbrace{\underline{b}}_{\substack{\perp \text{ to } \underline{v}. \\ \text{i.e. } \underline{b}\cdot\underline{v} = 0.}}$

$\underline{w}_{\parallel} \qquad \underline{w}_{\perp}$

how to find them:

$$\underline{w}_{\parallel} = \mathrm{proj}_{\underline{v}}(\underline{w}) = (\underline{w}\cdot\underline{e}_v)\,\underline{e}_v = \frac{\underline{w}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}\,\underline{v}.$$

$$\underline{w}_{\perp} = \underline{w} - \underline{w}_{\parallel}$$

**check** $\quad \underline{v}\cdot\underline{w}_{\perp} = \underline{v}\cdot(\underline{w} - \underline{w}_{\parallel}) = \underline{v}\cdot\underline{w} - \underline{v}\cdot\left(\dfrac{\underline{w}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}\right)\cdot\underline{v}.$

$$= \underline{v}\cdot\underline{w} - \underline{v}\cdot\underline{v}\left(\frac{\underline{w}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}\right) = \underline{v}\cdot\underline{w} - \underline{v}\cdot\underline{w} = 0$$

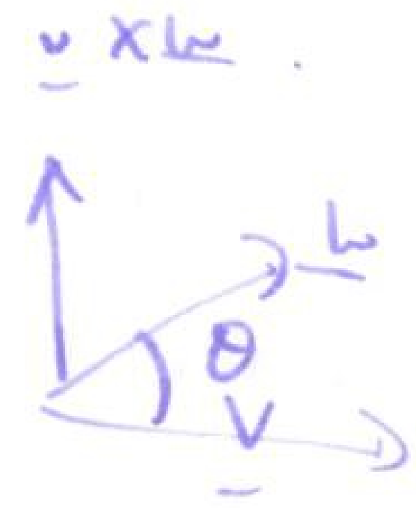
i.e. **can solve** $\quad \underline{v}\cdot(\underline{w} - \lambda\underline{v}) = 0.$

## §12.4 Cross product

special for $\mathbb{R}^3$.

**Def$^n$** $\quad \underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\underline{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\underline{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\underline{k}$

**Example** $\quad \underline{v} = \langle 1, 2, 3\rangle \qquad \underline{w} = \langle -1, 0, 3\rangle$.




**Th$^m$** $\underline{v}\times\underline{w}$ is the unique vector which

- is $\perp$ to both $\underline{v}$ and $\underline{w}$
- has length $\|\underline{v}\|\,\|\underline{w}\|\sin\theta$
- ~~st~~ $\{\underline{v}, \underline{w}, \underline{v}\times\underline{w}\}$ right-handed

**Example** $\quad \underline{i}\times\underline{j} = \underline{k}$

**Useful facts**

$\underline{w}\times\underline{v} = -\underline{v}\times\underline{w}$

$\underline{v}\times\underline{v} = \underline{0}$

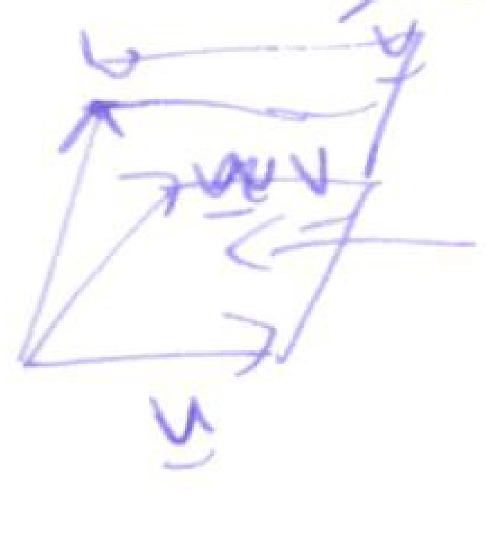
$\underline{v}\times\underline{w} = \underline{0}$ iff $\underline{v} = \lambda\underline{w}$ for $\lambda \neq 0$ (or one of $\underline{v}, \underline{w} = \underline{0}$).

$(\lambda\underline{v})\times\underline{w} = \underline{v}\times(\lambda\underline{w}) = \lambda(\underline{v}\times\underline{w})$.

$(\underline{u}+\underline{v})\times\underline{w} = \underline{u}\times\underline{w} + \underline{v}\times\underline{w}$

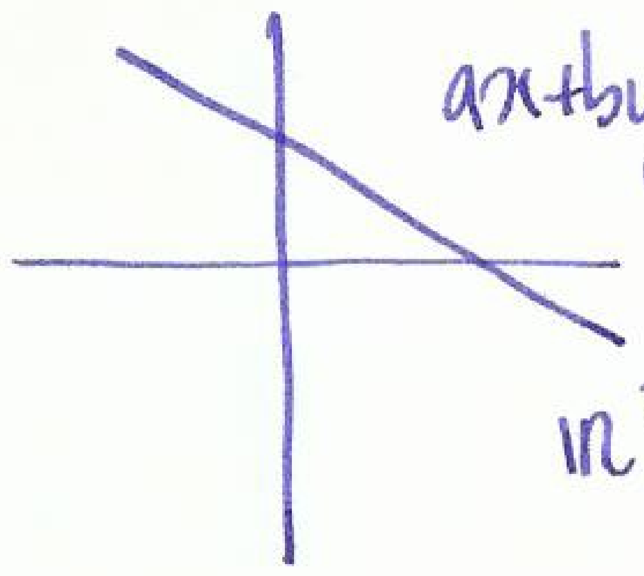
**Th$^m$**

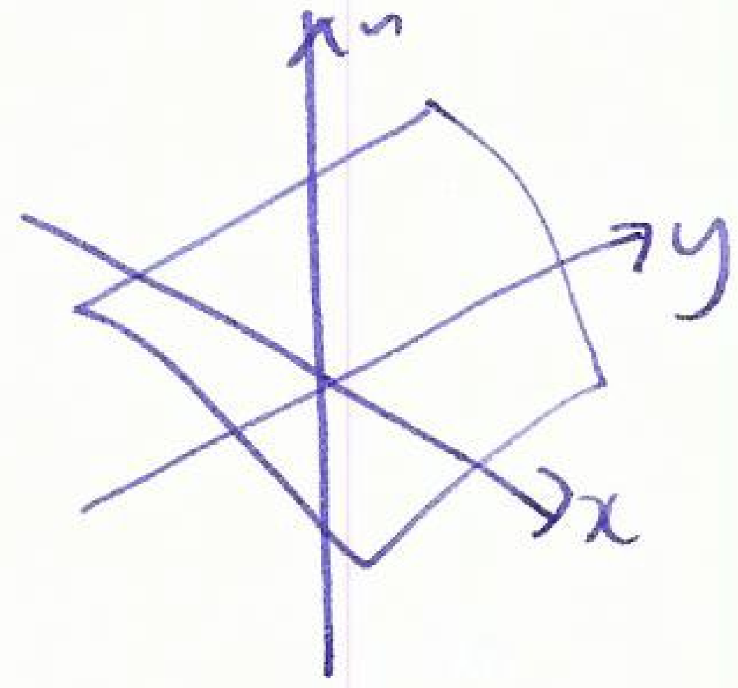

area of parallelogram $= \big|\, \|\underline{v}\times\underline{w}\| \,\big|$




volume of parallelepiped is $|\underline{u}\cdot(\underline{v}\times\underline{w})|$.

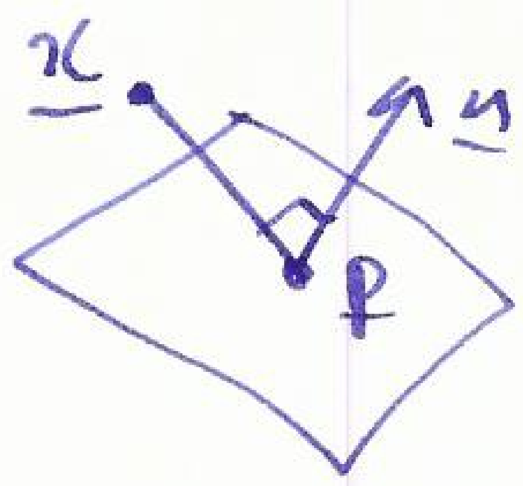
## §12.5 Planes in $\mathbb{R}^3$




$ax+by+cz=d$.
plane.

vector equation of plane




$\underline{n}\cdot(\underline{x}-\underline{p})=0$

$\underline{n}=\langle a,b,c\rangle$

$$\underline{n}\cdot(\underline{x}-\underline{p})=0$$

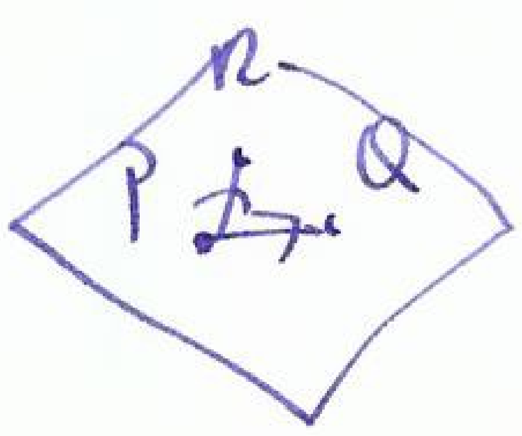
$$\underline{n}\cdot\underline{x}=\underline{n}\cdot\underline{p}$$

$$\langle a,b,c\rangle\cdot\langle x,y,z\rangle=\underline{n}\cdot\underline{p}.$$

$$ax+by+cz=\underline{n}\cdot\underline{p}$$

normal vector to $ax+by+cz=d$ is $\underline{n}=\langle a,b,c\rangle$.

3 points determine a plane




normal vector $\underline{n}=\overrightarrow{PQ}\times\overrightarrow{PR}$

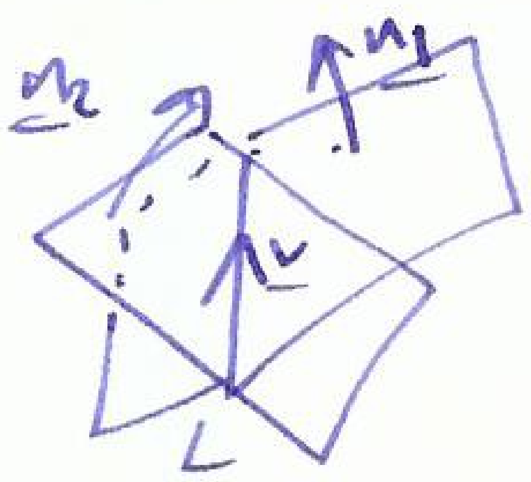
example $\quad P=(1,0,0)\quad \overrightarrow{PQ}=\langle -1,1,0\rangle$
$Q=(0,1,0)$
$R=(0,0,1)\quad \overrightarrow{PR}=\langle -1,0,1\rangle$.


$$\underline{n}=\begin{vmatrix}\underline{i}&\underline{j}&\underline{k}\\-1&1&0\\-1&0&1\end{vmatrix}=\underline{i}\begin{vmatrix}1&0\\0&1\end{vmatrix}-\underline{j}\begin{vmatrix}-1&0\\-1&1\end{vmatrix}+\underline{k}\begin{vmatrix}-1&1\\-1&0\end{vmatrix}.$$

$$=\underline{i}+\underline{j}+\underline{k}$$

$x+y+z=1$.

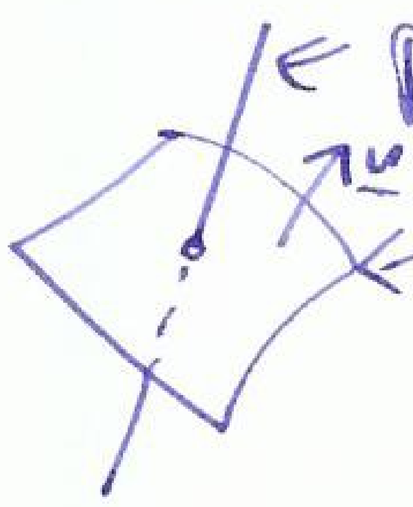
Intersection of two planes determines a line.



$\underline{v} = \underline{n}_1 \times \underline{n}_2$  to find point on $L$ solve

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2.$$

intersection of plane and line.



$\leftarrow$ parametric form $\underline{r}(t) = \underline{a} + \underline{v}t$

$\leftarrow \underline{n}\cdot(\underline{x}-\underline{p})=0$  $\left.\underline{n}\cdot(\underline{a}+\underline{v}t-\underline{p})=0\right\}$ solve for $t$.