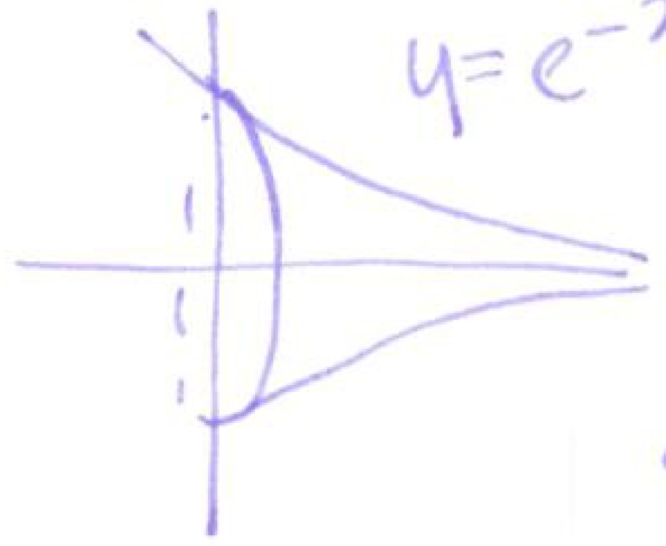
Example



$y = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$

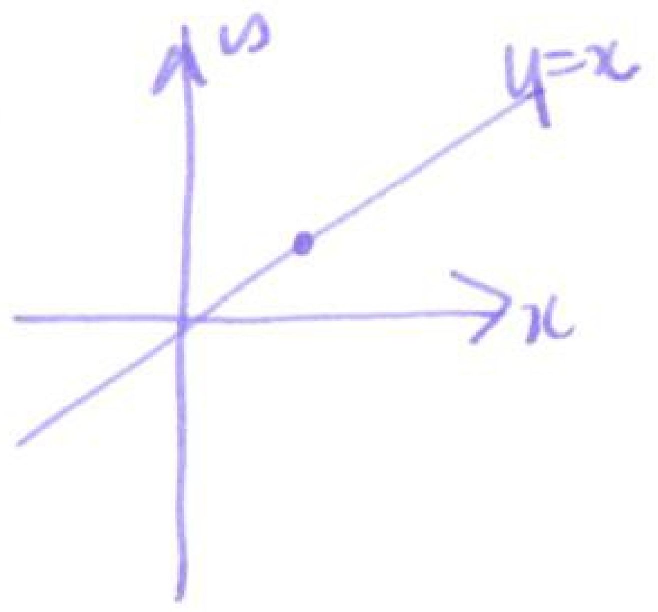
surface area $\displaystyle\int_0^\infty 2\pi e^{-x}\sqrt{1+(-e^{-x})^2}\,dx.$

sub $u = e^{-x}$

$\frac{du}{dx} = -e^{-x}$

$$\int_1^0 2\pi e^{-x}\sqrt{1+u^2}\,\underbrace{\frac{dx}{du}}_{=\frac{1}{-e^{-x}}}\,du = \int_1^0 -2\pi\sqrt{1+u^2}\,du. \quad \text{(trig integral)}.$$

§11.1 Parametric equations

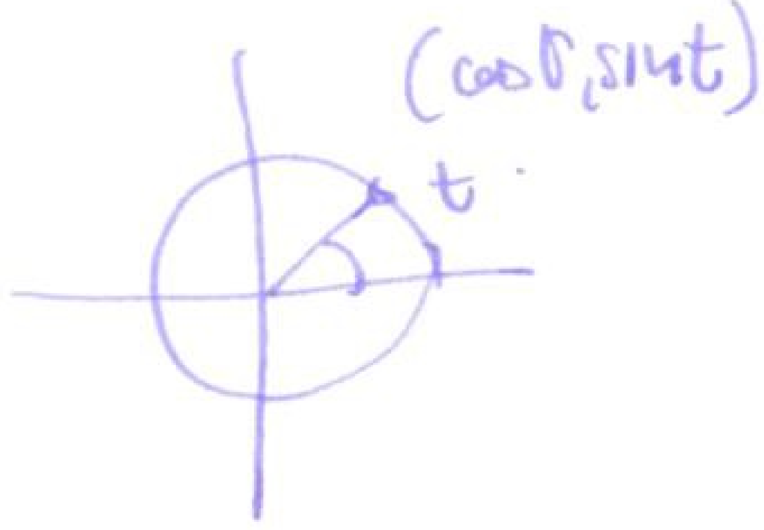


$x(t) = t$ $\quad c(t) = (x(t), y(t))$

$y(t) = t$

↑ much more flexible, eg.

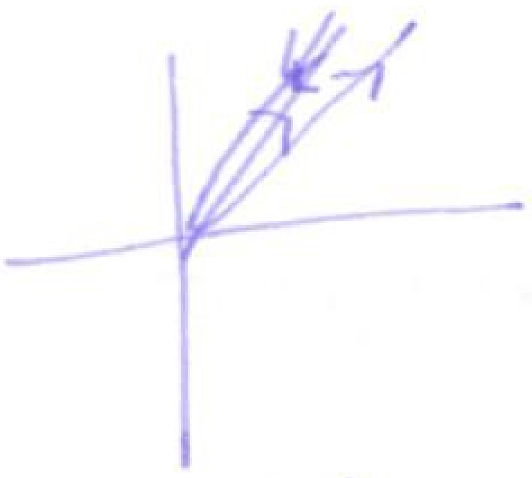
$c(t) = (\cos t, \sin t)$



Problem many parameterizations give the same curve,

eg $\quad c(t) = (2x(t), 2y(t)) \quad c(t) = ((x^{3}(t))^{3}, (y^{3}(t))^{3})$.

Warning: $c(t) = (x(t)^2, y(t)^2)$.

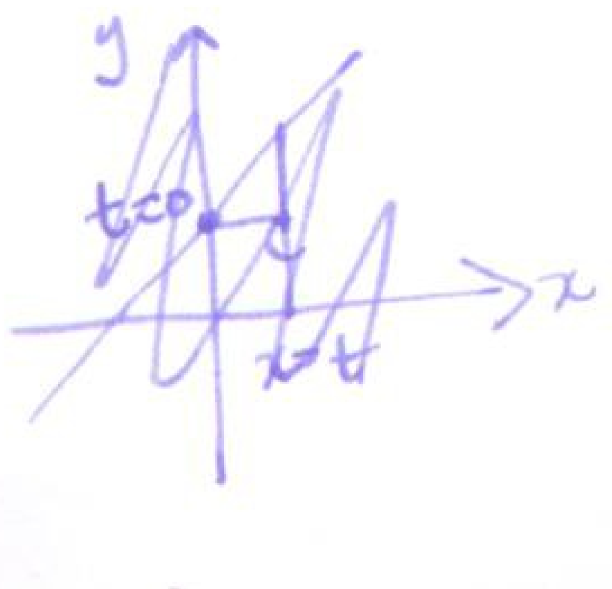


Example describe the parametric curve $c(t) = (t+1, t^2+1)$ in terms of $y=f(x)$.

$x = t+1 \Rightarrow t = x-1$

$y = t^2+1 \quad y = (x-1)^2+1 = x^2-2x+2$

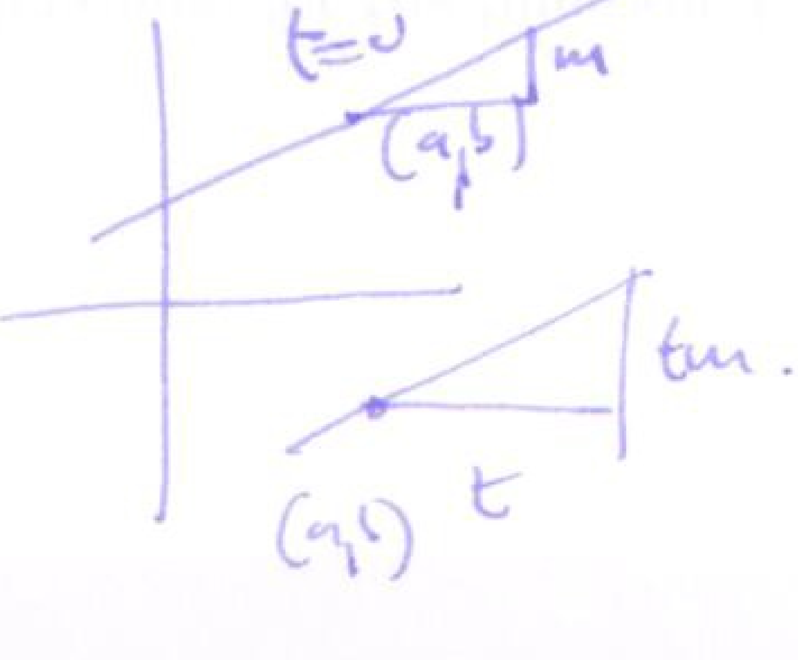
Example find a parametric equation for the line, through (a,b) with slope m and ~~y-intercept~~



~~$t=0$ $(0,c)$~~

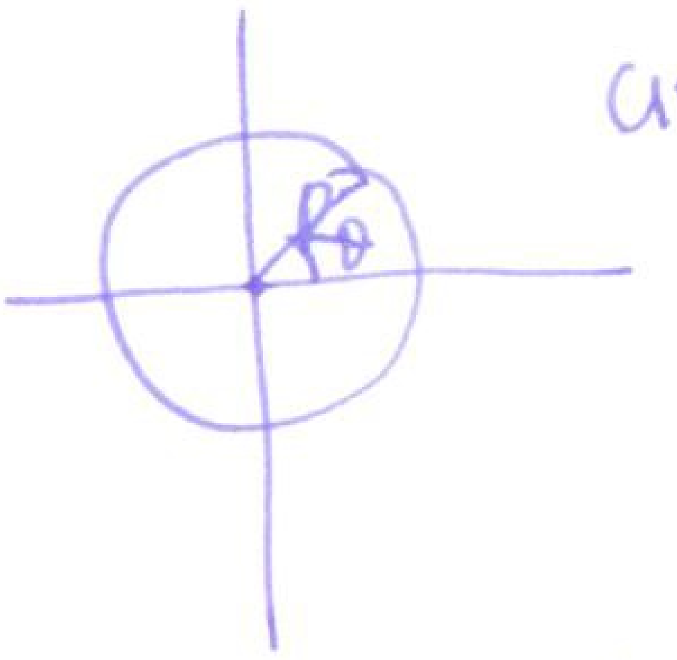
~~$x=t$ $y=mt+c$~~

~~$c(t) = ($~~



$t=0: (a,b)$.

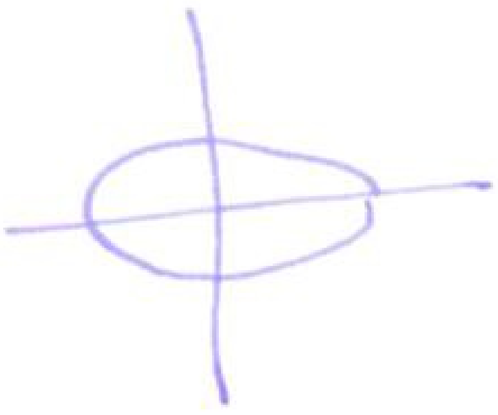
$c(t) = (a+t, b+mt)$.



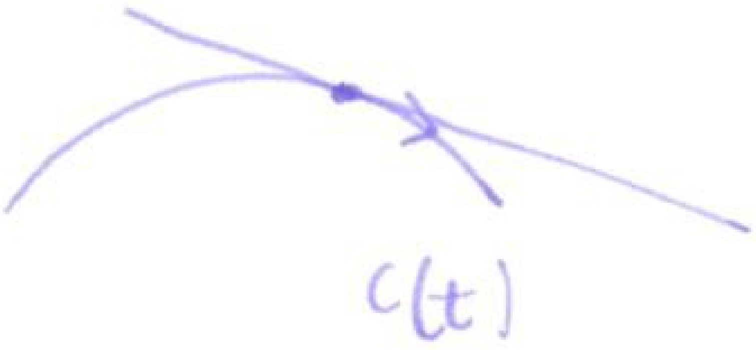
circle of radius R

$c(t) = (R\cos\theta, R\sin\theta)$

$c(t) = (R\cos\theta + a, R\sin\theta + b)$.

ellipse:  $c(t) = (R\cos\theta, S\sin\theta)$.

Q: how do find slope of tangent line?



A: $\boxed{\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}}$

why: $c(t) = (x(t), y(t))$.

chain rule:

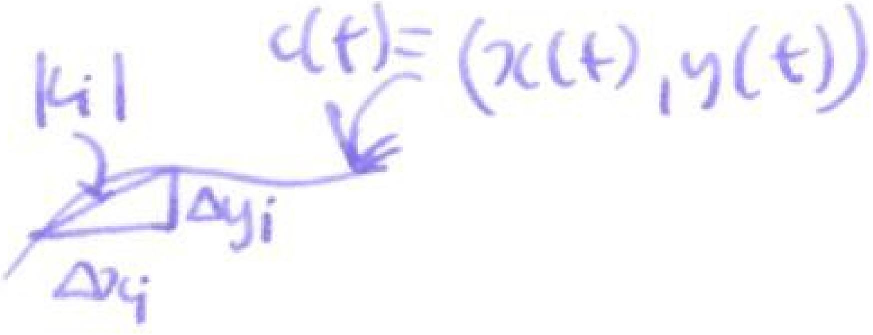
$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}.$$

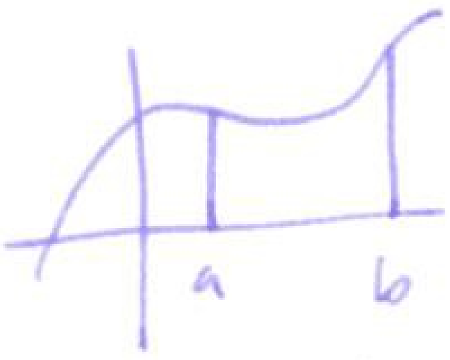
Example $c(t) = (\cos t, \sin 2t)$ $\quad \dfrac{dy}{dx} = \dfrac{2\cos 2t}{-\sin t}$

$x(t) = \cos t \quad \dfrac{dx}{dt} = -\sin t$

$y(t) = \sin 2t \quad \dfrac{dy}{dt} = 2\cos 2t$

§11.2 Arc length and speed



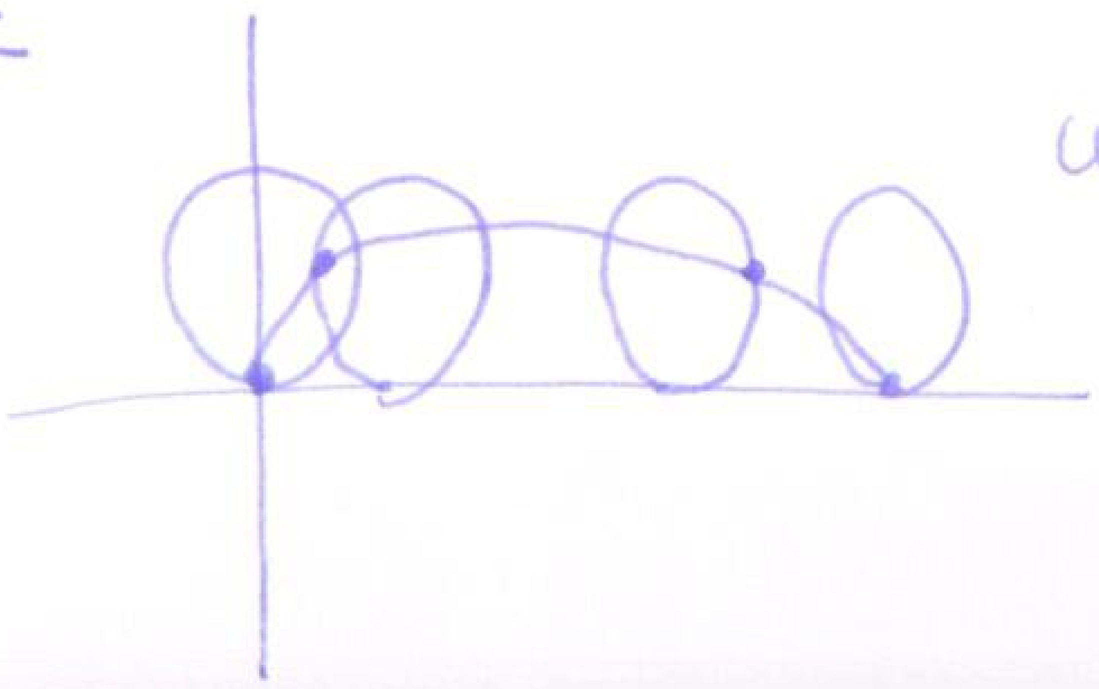
recall 

arc length $= \displaystyle\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

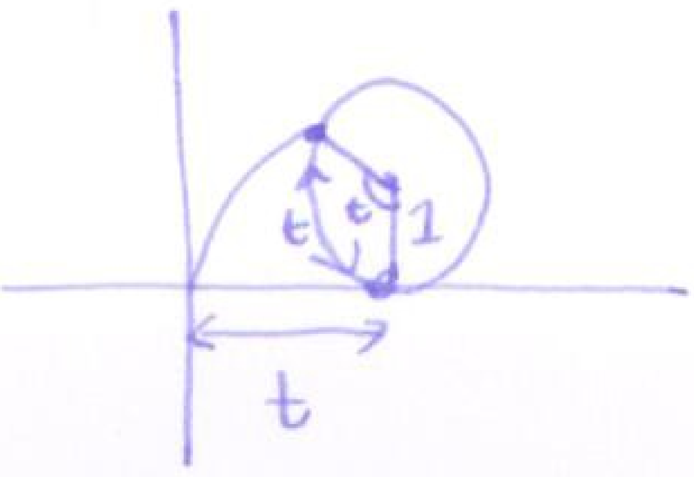
$$\text{arc length} = \sum_{i=1}^n \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_{i=1}^n \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\, \Delta t_i$$

take limit as $|\Delta x_i| \to 0$ $\quad$ arc length $= \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Example



cycloid.



$x(t) = t - \sin(t)$

$y(t) = 1 - \cos(t)$

$$\frac{dx}{dt} = 1-\cos(t) \qquad \frac{dy}{dt} = \sin(t).$$

arc length $\displaystyle\int_0^{2\pi} \sqrt{(1-\cos(t))^2 + (\sin(t))^2}\, dt = \int_0^{2\pi} \sqrt{1-2\cos t + \cos^2 t + \sin^2 t}\, dt$

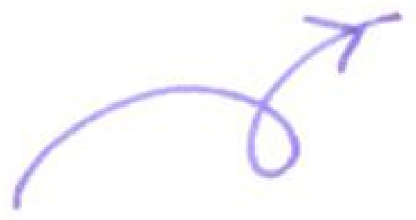
$$= \int_0^{2\pi} \sqrt{2-2\cos t}\, dt$$

recall: ~~cos~~ $\cos 2\theta = \cos^2\theta - \sin^2\theta$.
$= 1 - 2\sin^2\theta$.

$\cos t = 1 - 2\sin^2\left(\frac{t}{2}\right)$.

$$= \int_0^{2\pi} \sqrt{2 - 2\left(1-2\sin^2\left(\tfrac{t}{2}\right)\right)}\, dt.$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\left(\tfrac{t}{2}\right)}\, dt = \int_0^{2\pi} 2\left|\sin\left(\tfrac{t}{2}\right)\right| dt = \left[-4\cos\left(\tfrac{t}{2}\right)\right]_0^{2\pi} = 8.$$

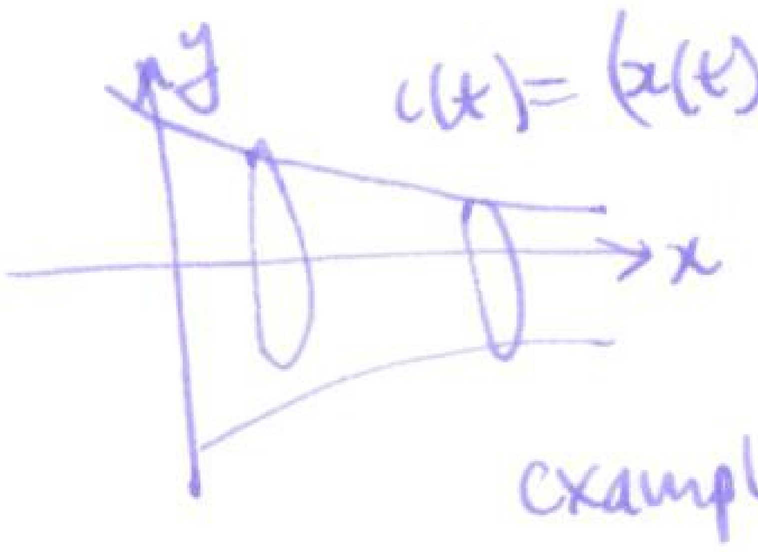


$c(t) = (x(t), y(t))$ speed $= \frac{d}{dt}(\text{arc length})$.

$$= \frac{d}{dt}\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\Bigg|_t.$$

surface area in parameterization

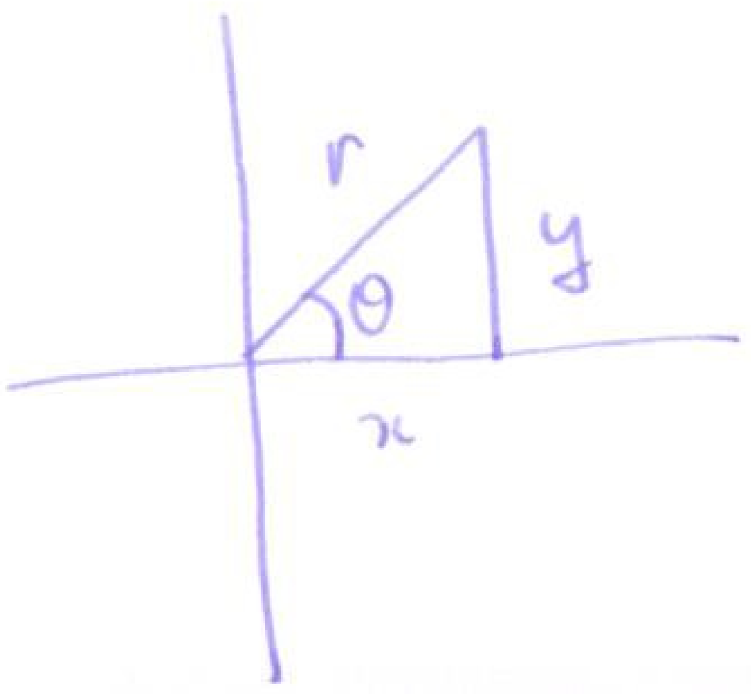


$c(t) = (x(t), y(t))$

surface area $\displaystyle = 2\pi \int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

example $c(t) = (t - \tanh t, \operatorname{sech} t)$

§11.3 Polar coordinates



$x = r\cos\theta$
$y = r\sin\theta$

~~θ = tan~~ $\tan\theta = y/x$

$r^2 = x^2 + y^2$.