Thm Root test: Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ (assume this limit exists).

then
- if $L < 1$ $\sum a_n$ converges absolutely
- if $L > 1$ $\sum a_n$ diverges
- if $L = 1$ no information.

Example $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$.

§10.6 Power series

Defn A power series centered at $x = c$ is an infinite sum of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

note: this gives a function of x, if the series converges.
series always converges for $x = c$! $F(c) = a_0$.

Thm Radius of convergence. Let $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ then

1. $F(x)$ converges only for $x = c$. $(R = 0)$
2. $F(x)$ converges for all x. $(R = \infty)$
3. there is a number $R > 0$ s.t. $F(x)$ converges absolutely if $|x-c| < R$ and diverges if $|x-c| > R$. It may or may not converge for $x = c \pm R$.

We call R the radius of convergence.

Example Q: for what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge?

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{2}\right|$

so converge for $\left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$. (so $R = 2$). can check endpoints...

Example $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-5)^n$. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x-5)^{n+1}}{n+1} \cdot \frac{n}{(-1)^n (x-5)^n}\right|$.

$= \lim_{n\to\infty} \left|(x-5)\frac{n}{n+1}\right| = x-5$. So converges for $|x-5| < 1$.

Examples $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sum_{n=0}^{\infty} n!\, x^n$.

Thm Suppose $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has radius of convergence $R>0$

then $F(x)$ is differentiable on $(-R-c, c+R)$ and furthermore:

$$F'(x) = \sum_{n=1}^{\infty} a_n n (x-c)^n$$

$$\int F(x)\,dx = c_1 + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n}$$

Example ① $1+x+x^2+x^3+\cdots = \frac{1}{1-x}$. radius of converge $R=1$.

so $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots$ } radius of convergence $= 1$.

and $-\ln|1-x| \cancel{+c} = c + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$ }

② $1-x^2+x^4 \cancel{-4} x^6 + \cdots = \frac{1}{1+x^2}$ radius of converge $R=1$.

so $\tan^{-1}(x) \cancel{+c_1} = c + x - \frac{x^3}{3} + \frac{x^5}{5} \cancel{+} - \frac{x^7}{7} + \cdots$

③. Fact $e^x + c$ is the (unique) solution to $\frac{dy}{dx} = y$ $f'(x) = f(x)$.

$$\frac{d}{dx}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

so $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ radius of convergence $R=\infty$.

How to find a power series solution of a differential equation:

try: $y = a_0 + a_1 x + a_2 x^2 + \cdots$ ~~and~~ (and $y(0)=1$).

Example $y' = y$ $a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \underset{1}{a_0} + a_1 x + a_2 x^2 + \cdots$

~~$a_1 = a_0$~~ $a_1 = 1$

$2a_2 = a_1$ $a_2 = \frac{1}{2}$.

$3a_3 = a_2$ $a_3 = \frac{1}{2\cdot 3} = \frac{1}{3!}$ etc.

$y'' + y' + y = 0$ $\quad$ $1 + a_1 x + a_2 x^2 + \cdots$ $\quad$ $2 + 2a_2 = 0$

$y'(0) = 1$ $\quad y(0) = 1$. $\quad$ $1a_1 + 2a_2 x + 3a_3 x^2 + \cdots$ $\quad$ $a_2 = -1$.

$y''(0) = 1$ $\quad$ $2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots$

$2 + 2a_2 = 0$ $\quad a_2 = -1$

$1 + 2a_2 + 6a_3 = 0$ $\quad a_3 = \frac{-1+2}{6} = \frac{1}{6}$.

$a_2 + 3a_3 + 12a_4 = 0$ $\quad a_4 = \cdots$

§10.7 Taylor series

spse $f(x)$ has a power series expansion at $x = c$, i.e.

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

Q: how do we find the a_i.

A: $f(c) = a_0$

$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$

$f'(c) = a_1$.

$f''(c) = 2a_2 + 6a_3(x-c) + \cdots$ $\quad$ $f''(c) = 2a_2$.

so $\boxed{a_n = \frac{f^{(n)}(c)}{n!}}$

Example e^x, $\sin(x)$, $\cos(x)$, x^n centered at 0.

Thm If $f(x)$ is represented by a power series at $x = c$ with radius of convergence $R > 0$, then $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, where $a_n = \frac{f^{(n)}(c)}{n!}$.

Useful facts If two Taylor series are convergent then can

- differentiate
- integrate
- multiply $f(x)g(x)$.
- substitute $f(g(x))$.

Examples xe^x, $e^x \sin(x)$

e^{-x^2}