Example does $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2-9}}$ converge? compare with $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2-9}}}{\frac{1}{n}} = \frac{n}{\sqrt{n^2-9}} = \frac{1}{\sqrt{1-9/n^2}} = 1.$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge

§10.4 Absolute and conditional convergence

Q: what about $\frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$? (*)

Defn A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

So (*) is absolutely convergent.

Example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ not absolutely convergent.

Thm Absolute convergence $\Rightarrow$ convergence.

Proof $0 \leq a_n + |a_n| \leq 2|a_n|$

$2\sum_{n=1}^{\infty} |a_n|$ converges. $\Rightarrow$ $\sum_{n=1}^{\infty} a_n + |a_n|$ converges.

but then $\sum_{n=1}^{\infty} a_n + |a_n| - |a_n| = \sum_{n=1}^{\infty} a_n + |a_n| - \sum_{n=1}^{\infty} |a_n|$

converge. $\Leftarrow$ converges converge

$\sum_{n=1}^{\infty} a_n$ converges $\square$.

Example does $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converge absolutely?

apply integral test to $\sum \frac{1}{n \ln n}$ A: no.

Q: does $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converge A: dunno.

Defn conditional convergence: $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ does not converge.

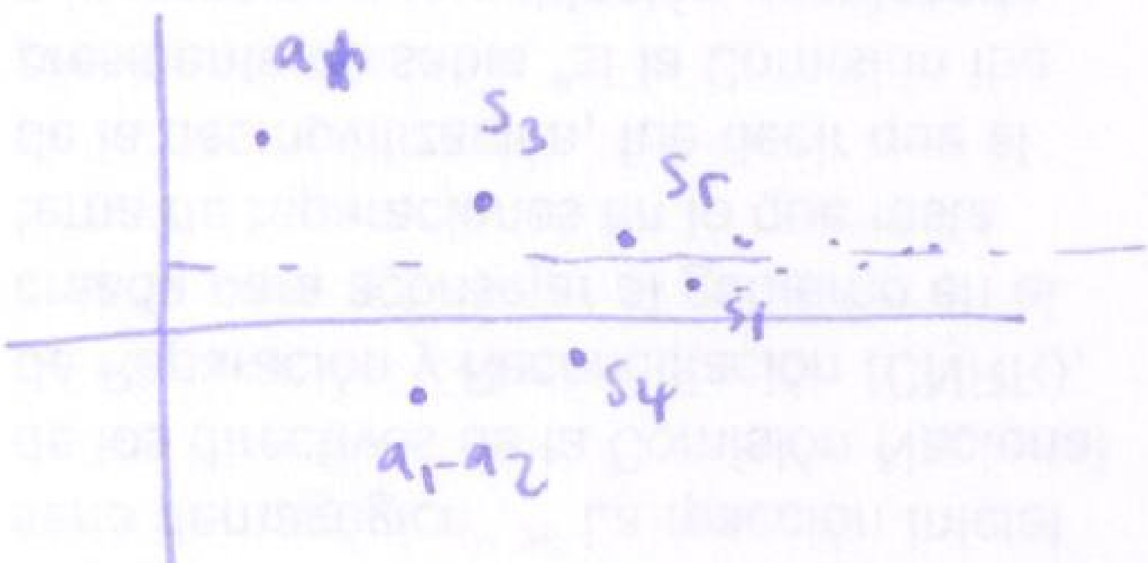
Thm Alternating series test

Let $\{a_n\}$ be decreasing, positive sequence $a_n \to 0$.

Then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges. $= S$ Furthermore $0 \leq S \leq a_1$

and $S_{2n} \leq S \leq S_{2n+1}$ for all n.

Proof



even partial sums $S_{2n} = \underbrace{a_1 - a_2}_{>0} + \underbrace{a_3 - a_4}_{>0} + \cdots + \underbrace{a_{2n-1} - a_{2n}}_{>0}$

positive increasing sequence

odd partial sums $S_{2n+1} = a_1 - \underbrace{(a_2 - a_3)}_{<0} - \underbrace{(a_4 - a_5)}_{<0} - \cdots - \underbrace{(a_{2n} - a_{2n+1})}_{<0}$

decreasing sequence.

furthermore $S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - a_{2n}$ so $S_{2n} \leq a_1$ for all n.

so S_{2n} is a positive increasing sequence, so $\lim_{n \to \infty} S_{2n}$ exists.

similarly $\lim_{n \to \infty} S_{2n+1}$ exists.

but now $\lim_{n \to \infty} S_{2n} - S_{2n+1} = \lim_{n \to \infty} S_{2n} - \lim_{n \to \infty} S_{2n+1}$

$\qquad = \lim_{n \to \infty} -a_{2n+1} = 0$ $\square$.

Example show $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$ converges (alternating harmonic series).

use alternating series test: set $\{a_n\}$ $a_n = \frac{1}{n}$

then a_n positive decreasing sequence and $a_n \to 0$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges $\square$.

so $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}$ is not absolutely convergent, but converges (conditionally convergent).

§10.5 Ratio and root tests.

fact $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ ← Q: how do we show that this converges.

(A: comparison test $n! = 1\cdot 2\cdot 3\cdot 4 \cdots (n-2)(n-1)(n) > (n-1)^2$ so $\frac{1}{n!} < \frac{1}{(n-1)^2}$)

Thm Ratio test $\{a_n\}$ sequence, and suppose $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ limit exists.

Then: 1 if $\rho < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely

2 if $\rho > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges

3 if $\rho = 1$ no information.

Proof If $\rho < 1$ then there is a number $\rho < r < 1$

$\left|\frac{a_{n+1}}{a_n}\right| \to \rho$ so there is an N s.t. $\left|\frac{a_{n+1}}{a_n}\right| < r$ for all $n \geq M$

so $|a_{M+1}| < r\,|a_M|$

$|a_{M+2}| < r\,|a_{M+1}| < r^2\,|a_M|$

$|a_{M+3}| < r\,|a_{M+2}| < r^3\,|a_M|$

so $\sum_{n=M}^{\infty} |a_n| \leq \sum_{n=0}^{\infty} |a_M| r^n \leq \frac{|a_M|}{1-r}$ so converges by comparison test if $\rho > 1$ then choose r s.t. $\rho > r > 1$

then as $\left|\frac{a_{n+1}}{a_n}\right| \to \rho$ there is an M s.t. $\left|\frac{a_{n+1}}{a_n}\right| > r > 1$ for all $n \geq M$

then $|a_{M+1}| > r|a_M|$

$|a_{M+2}| > r^2|a_M|$ so $\sum_{n=M}^{\infty} |a_n| \geq |a_M| \sum_{n=1}^{\infty} r^n$ diverges.

etc. by comparison test. $\square$

Example show $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)!}{1/n!}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$

Example show $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges.

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}}\right|.$

$= \lim_{n\to\infty} \left| (1+1/n)^3 \frac{1}{3} \right| = \frac{1}{3} < 1.$

Bad example

$\sum_{n=1}^{\infty} n^2 \qquad \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1.$

$\sum_{n=1}^{\infty} n^{-2} \qquad \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = 1.$

Th^{m} (Root test) $\{a_n\}$ sequence, suppose that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists.

① if $L < 1$ then $\sum a_n$ converges absolutely

② if $L > 1$ then $\sum a_n$ diverges

③ if $L = 1$ no information.

Example $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$.