Example $s = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots \qquad f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}.$

$$\int_1^\infty x^{-1/2}\,dx = \lim_{N\to\infty} \int_1^N x^{-1/2}\,dx = \lim_{N\to\infty} \left[2x^{1/2} \right]_1^N = \lim_{N\to\infty} 2\sqrt{N} - 2 \to \infty$$

so S diverges.

Thm Convergence of p-series. $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$.

Proof (integral test) $\sum_{n=1}^\infty a_n$, $a_n = f(n)$ $\quad f(x) = \frac{1}{x^p} = x^{-p}$

$$\lim_{N\to\infty} \int_1^N x^{-p}\,dx = \lim_{N\to\infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^N = \cancel{N^{-p+1}} \quad \frac{N^{-p+1}}{p+1}$$

converges if $-p+1 < 0$, $p > 1$. diverges otherwise.

Example

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots = \sum_{n=1}^\infty \frac{1}{n^{1/2}}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots = \sum_{n=1}^\infty \frac{1}{n^3} = ?$$

Thm Comparison test: suppose $0 \leq a_n \leq b_n$ for all $n \geq M$.

then ① if $\sum_{n=1}^\infty b_n$ converges then $\sum_{n=1}^\infty a_n$ also converges

② if $\sum_{n=1}^\infty a_n$ diverges $\Rightarrow \sum_{n=1}^\infty b_n$ diverges.

Example $S = \sum_{n=1}^\infty 2^{-n^2} = 2^{-1} + 2^{-4} + 2^{-9} + \cdots$

$$\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^9} + \cdots$$

note $\frac{1}{2^{n^2}} < \frac{1}{2^n}$ geometric series, converges.

Thm limit comparison test $\{a_n\}, \{b_n\}$ positive sequences.

suppose $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists.

Then: • if $L > 0$ $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges

• $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof (sketch)

case $L > 0$ then $\frac{a_n}{b_n} \to L$ so $0 < \frac{a_n}{b_n} < R$ $\quad L < R.$

$0 < a_n < R b_n$

comparison test.

so b_n converges $\Rightarrow a_n$ converges

similarly $\frac{b_n}{a_n} \to \frac{1}{L}$ so $0 < \frac{b_n}{a_n} < R'$ $\quad \frac{1}{L} < R'$

$0 < b_n < R' a_n$

so comparison test means a_n converges $\Rightarrow b_n$ converges.

note if $L = 0$, only get one direction.... $\square$

Example show $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n}$ converges. note for large n, $a_n \sim n^{-2}$

compare with $b_n = n^{-2}$ $\quad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n^2}{n^4-n}}{\frac{1}{n^2}} = \frac{n^4}{n^4 - n} = \frac{1}{1 - 1/n^3} = 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series) $\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{n^4-n}$ converges