Geometric series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} \qquad a_n = \frac{1}{2^n}$$

$$S_N = \frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^N}$$

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4} = 1-\frac{1}{4}.$$

$$S_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8} = 1-\frac{1}{8}$$

$$S_4 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{15}{16} = 1-\frac{1}{16}$$

$$\cancel{S_{N+1} = \frac{1}{2}+\frac{1}{4}+\cdots}$$

$$\frac{1}{2}S_N = \frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{N+1}}$$

$$S_N - \frac{1}{2}S_N = \frac{1}{2}-\frac{1}{2^{N+1}}$$

$$S_N\left(\frac{1}{2}\right) = \frac{1}{2}-\frac{1}{2^{N+1}}$$

$$S_N = 1-\frac{1}{2^N}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1-\frac{1}{2^N} = 1.$$

General case: $\quad c+cr+cr^2+cr^3+\cdots = \sum_{n=0}^{\infty} cr^n$

$$S_N = c+cr+cr^2+\cdots+cr^N$$

$$rS_N = cr+cr^2+cr^3+\cdots+cr^{N+1}$$

$$S_N - rS_N = c-cr^{N+1} = c(1-r^{N+1}).$$

$$S_N(1-r) = c(1-r^{N+1}).$$

$$S_N = \frac{c(1-r^{N+1})}{1-r} \qquad \lim_{N\to\infty}\frac{c(1-r^{N+1})}{1-r} = \frac{c}{1-r} \text{ if } |r|<1.$$

otherwise doesn't converge.

Special case (Telescoping)

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

$$\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1}$$

$$S_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{1}{1} - \frac{1}{3}$$

$$S_3 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = \frac{1}{1} - \frac{1}{4}.$$

etc. $S_N = 1 - \frac{1}{N+1}$ $\lim_{N\to\infty} S_N = 1.$

Example $\underbrace{1}_{\geq 1} + \underbrace{\frac{1}{2} + \frac{1}{3}}_{\geq 1} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 1.} + \cdots$ diverges.

Q: when does a series converge? A: dunno.

Tools: ~~Thm~~ Divergence test. $\sum_{n=1}^{\infty} a_n$. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Warning If $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

Recall $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

Rules for infinite sums / aka series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series.

Then $\sum a_n + \sum b_n = \sum (a_n + b_n)$ $\sum c a_n = c \sum a_n$

$\sum a_n - \sum b_n = \sum (a_n - b_n)$

§10.3 Series with positive terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots \qquad \text{all } a_n \geq 0$$

note: the sequence S_N is an increasing sequence.

$$S_{N+1} = S_N + \underbrace{a_{N+1}}_{\geq 0}$$

Recall An increasing series with an upper bound converges.

Thm Let $S = \sum_{n=1}^{\infty} a_n$ be a positive series. Then exactly one of the following occurs: ① the partial sums S_N are bounded above, and S converges

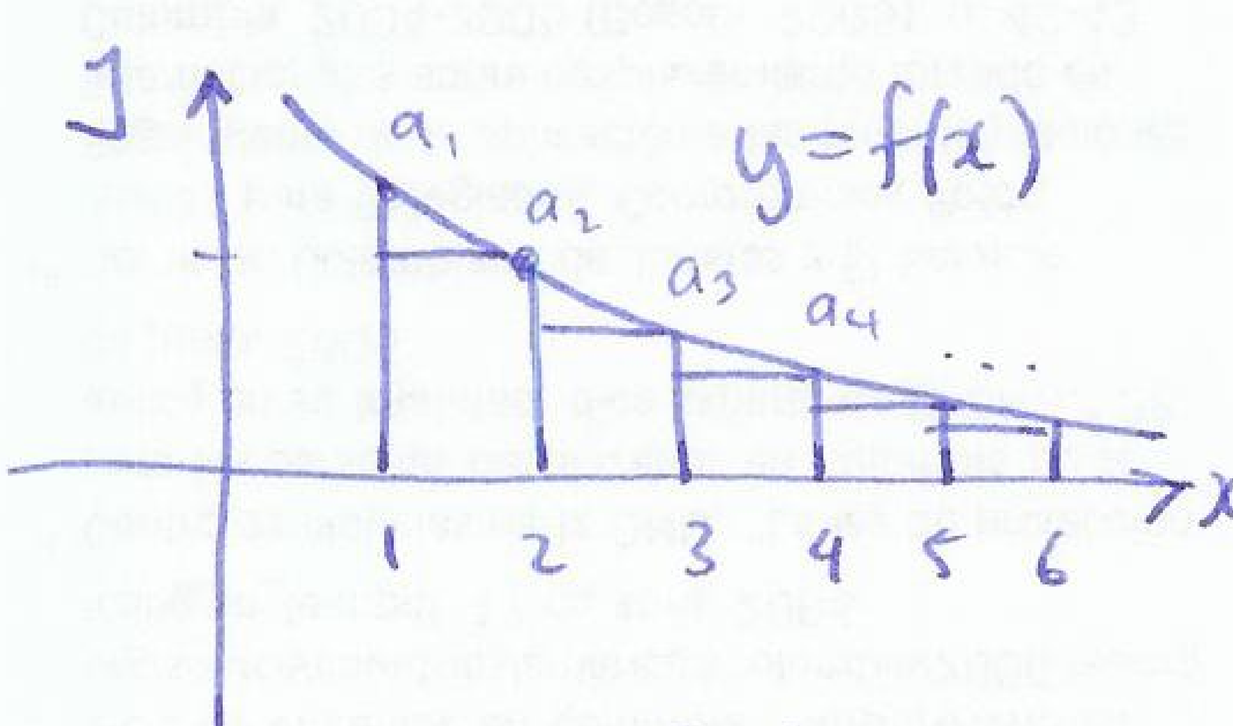
② the partial sums are not bounded above and S diverges.

Thm Integral test. Let $a_n = f(n)$ $f(x)$ positive, decreasing, continuous for $x \geq 1$.

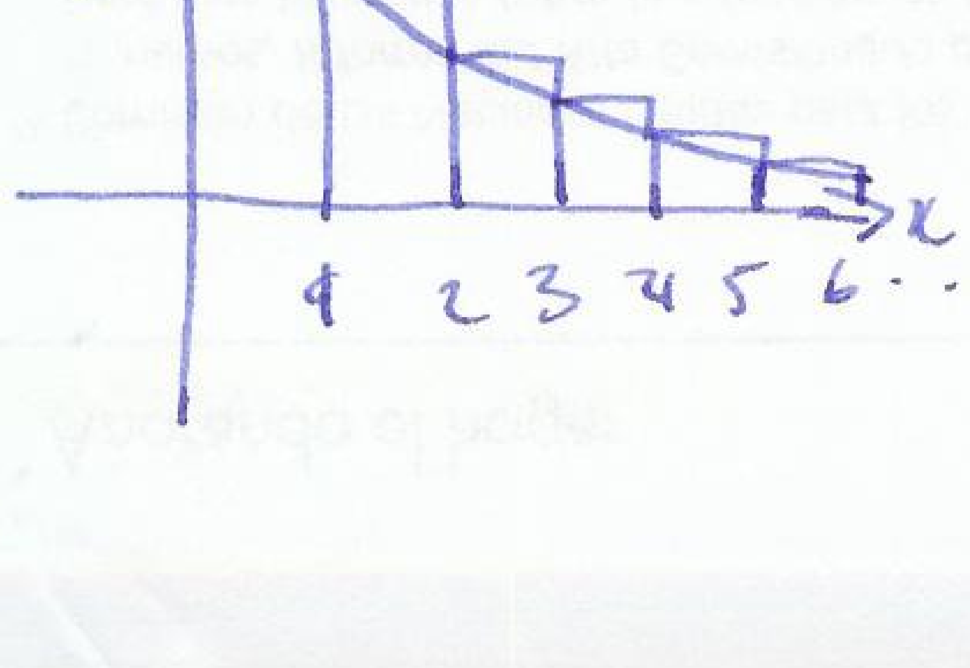
then ① if $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

② if $\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

Proof



$$S_N - a_1 = a_2 + a_3 + a_4 + \dots a_N \leq \int_1^N f(x)\,dx.$$



$$S_N = a_1 + a_2 + a_3 + \cdots + a_N \geq \int_1^N f(x)\,dx.$$

□.