Example $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ $a_n = \frac{n-1}{n}$ $f(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$.

$$\lim_{x \to \infty} 1 - \frac{1}{x} = 1 \Rightarrow \lim_{n \to \infty} \frac{n-1}{n} = 1.$$

Example (Geometric sequences) $a_n = r^n$

e.g. $2, 4, 8, 16, 32, \ldots$ $a_n = 2^n$

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$ $a_n = \frac{1}{3^n}$

$1, 1, 1, 1, \ldots$ $a_n = 1^n$

Fact
$$\lim_{n \to \infty} r^n = \begin{cases} \infty & r > 1 \\ 1 & r = 1 \\ 0 & |r| < 1 \end{cases}$$

Rules for limits of sequences: same as rules for limits of functions.

Assume $a_n \to L$, $b_n \to M$ then

- $\lim_{n \to \infty} a_n + b_n = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = L + M$
- $\lim_{n \to \infty} a_n b_n = \left(\lim_{n \to \infty} a_n\right)\left(\lim_{n \to \infty} b_n\right) = LM$
- $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} = \frac{L}{M}$
- $\lim_{n \to \infty} c a_n = c \lim_{n \to \infty} a_n = cL$ c constant.

Squeeze Thm If $a_n \le b_n \le c_n$ and $\lim_{n \to \infty} a_n = L$, $\lim_{n \to \infty} c_n = L$

then $\lim_{n \to \infty} b_n = L$.

Example $\lim\limits_{n\to\infty} \dfrac{R^n}{n!} = 0$ for any R.

note there is an integer M s.t. $M \le R \le M+1$

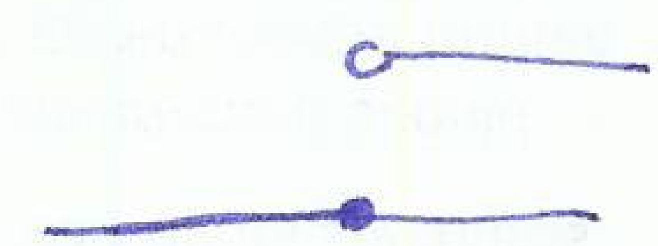
$$0 \le \frac{R^n}{n!} = \underbrace{\frac{R}{1}\cdot\frac{R}{2}\cdots\cdots\frac{R}{M}}_{\text{call this } C}\cdot\underbrace{\frac{R}{M+1}\cdot\frac{R}{M+2}}_{\le 1.}\cdots\cdots\frac{R}{n} \le C\frac{R}{n}$$

but $\lim\limits_{n\to\infty} C\dfrac{R}{n} = 0.$ So $\lim\limits_{n\to\infty} \dfrac{R^n}{n!} = 0.$

Th^m If $f(x)$ is continuous and $\lim\limits_{n\to\infty} a_n = L$,

then $\lim\limits_{n\to\infty} f(a_n) = f\left(\lim\limits_{n\to\infty} a_n\right) = f(L).$

Important f cts. Non-example: $f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$



then $\frac{1}{n} \longrightarrow 0$

$f(\frac{1}{n}) = 1$ for all n. but $f(0) = 0. \ne 1.$

Example compute $\lim\limits_{n\to\infty} e^{n/n+1}$ start with $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = \lim\limits_{n\to\infty} \dfrac{1}{1+1/n} = 1.$

so $\lim\limits_{n\to\infty} e^{n/n+1} = e^{\lim\limits_{n\to\infty} n/n+1} = e^1. = e.$

Def^n A sequence $\{a_n\}$ is

- bounded above if $a_n \le M$ for all n
- bounded below if $L \le a_n$ for all n
- bounded if $L \le a_n \le M$ for all n.

Th^m Convergent sequences are bounded.

Warning: Bounded sequences need not converge.

Example $a_n = 1, 0, 1, 0, 1, 0, \ldots$ $\qquad a_n = \dfrac{1 \mp (-1)^n}{2}$.

Thm: Bounded monotonic sequences converge.

- If $\{a_n\}$ is increasing and $a_n \leq M$ then $\lim_{n\to\infty} a_n$ exists and is $\leq M$
- If $\{a_n\}$ decreasing and $L \leq a_n$, then $\lim_{n\to\infty} a_n$ exists and is $\geq L$.

Example $a_n = \frac{1}{n}$. decreasing. want: $\frac{1}{n} > \frac{1}{n+1}$.

show: $n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1}$.

lower bound $L = -100$.

so $\lim_{n\to\infty} \frac{1}{n} \geq -100$.

Example Show $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing and bounded below.

try: $f(x) = \sqrt{x+1} - \sqrt{x} = (x+1)^{1/2} - x^{1/2} > 0$ so bounded below by 0.

$f'(x) = \frac{1}{2}(x+1)^{-1/2} - \frac{1}{2}x^{-1/2}$

$= \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}\right) < 0$

note: $x+1 > x$.

$\sqrt{x+1} > \sqrt{x}$.

$\frac{1}{\sqrt{x+1}} < \frac{1}{\sqrt{x}}$,

so $f'(x) < 0 \Rightarrow$ decreasing

§10.2 Series

Def$^{\underline{n}}$ A <u>series</u> is an infinite sum $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$

Example

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$1 + 1 + 1 + 1 + \cdots$

$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

$1 - 1 + 1 - 1 + 1 - 1 + 1 \cdots$

Def$^{\underline{n}}$ The n-th partial sum $S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n$

Def$^{\underline{n}}$ The sum of the infinite series is defined to be the limit of the partial sums, <u>if this limit exists</u>. $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$

If $\lim_{N\to\infty} S_N = S$ then $\sum_{n=1}^{\infty} a_n$ <u>converges</u> and we write $\sum_{n=1}^{\infty} a_n = S$.

Example ① $1 + 1 + 1 + 1 + \cdots$

$S_N = \underbrace{1 + 1 + \cdots + 1}_{N\text{-times}} = N$ $\qquad \lim_{N\to\infty} N = \infty$. $\quad$ so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1$ does <u>not</u> converge.

② $1 - 1 + 1 - 1 + 1 - 1 \cdots$

$S_1 = 1$

$S_2 = 1 - 1 = 0$

$S_3 = 1 - 1 + 1 = 1$

$\vdots$

$S_n = \frac{1 - (-1)^n}{2}$

$(S_n) = 1, 0, 1, 0, 1, 0, 1, \ldots$ <u>does not converge</u>.