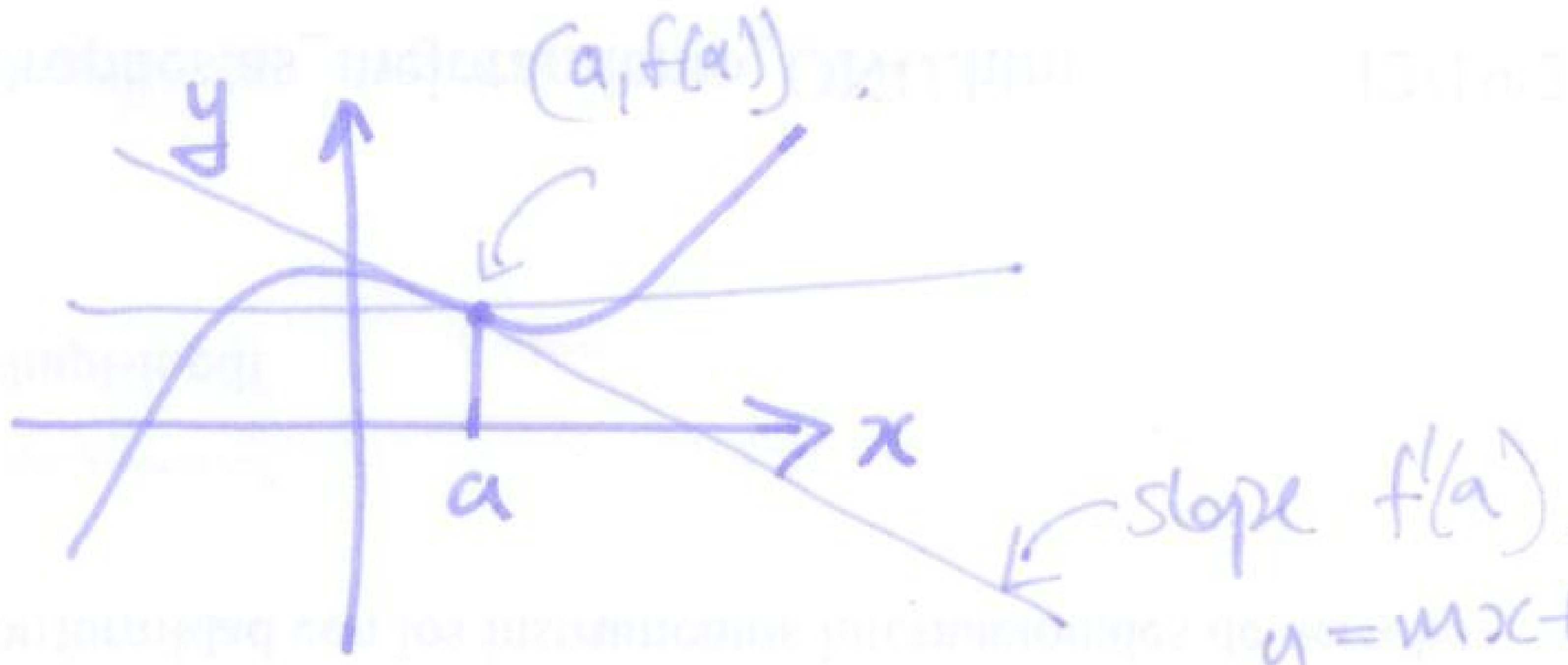
§8.4 Taylor polynomial



Approximations to functions:

$y = mx + C$

0-th approx. $f(x) \simeq f(a)$

1-st approx (straight line) $f(x) \simeq f(a) + f'(a)(x-a)$ ← $y - y_0 = m(x - x_0)$.

$y - f(a) = f'(a)(x-a)$.

2nd approx (quadratic) } how do we find these?

3rd approx (cubic) } · choose quadratic whose first 2 derivatives agree with $f(x)$ at a.

etc. ⋮ · choose cubic whose first 3 derivatives agree with $f(x)$ at a.

quadratic: $T_2(x) = a_0 + a_1(x-a) + a_2(x-a)^2$

deg 0: $x=a$: $f(a) = a_0 + 0 + 0 = f(a)$. so $a_0 = f(a)$

deg 1: $x=a$: $f'(x) = a_1 + 2a_2(x-a)$

$f'(a) = a_1 = f'(a)$ so $a_1 = f'(a)$

deg 2: $f''(x) = 2a_2 = f''(a)$ so $a_2 = \frac{1}{2} f''(a)$.

so $T_2(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2$.

general $T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n$.

differentiate k times $f^{(k)}(x) = k!\,a_k + (k+1)!\,a_{k+1}(x-a) + \cdots + \frac{n!}{(n-k)!}(x-a)^{n-k}$.

$f^{(k)}(a) = k!\,a_k$ so $a_k = \frac{1}{k!} f^{(k)}(a)$.

so

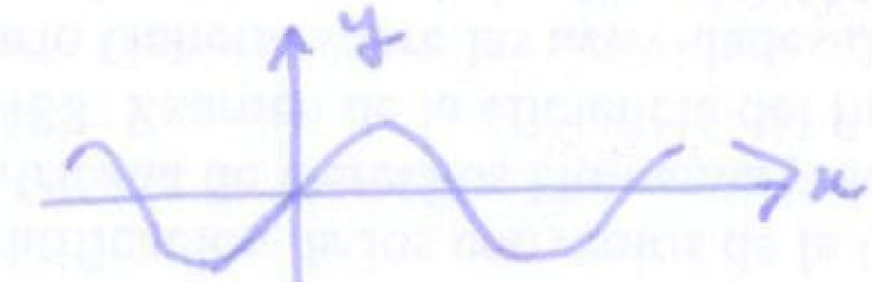
$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n$

at a.

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Example $y = \sin(x)$

at $x = 0$.



$f(x) = \sin(x)$ $\quad f(0) = 0$ so

$f'(x) = \cos(x)$ $\quad f'(0) = 1$

$f''(x) = -\sin(x)$ $\quad f''(0) = 0$

$f^{(3)}(x) = -\cos(x)$ $\quad f^{(3)}(0) = -1$

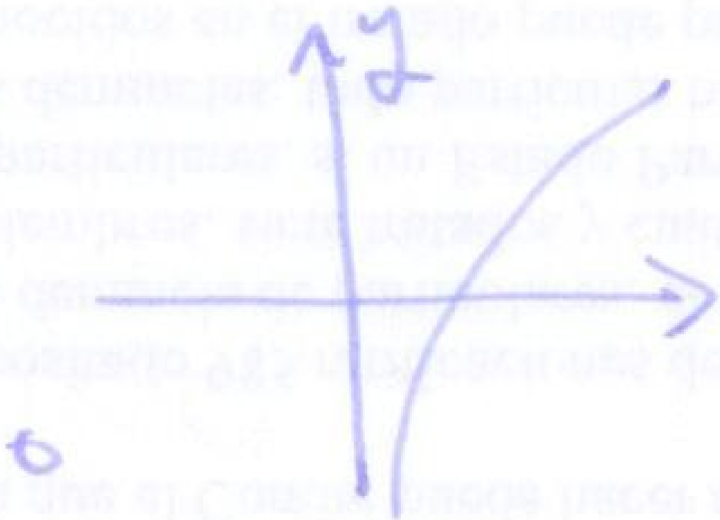
$f^{(4)}(x) = \sin(x)$ $\quad f^{(4)}(0) = 0$.

$T_n(x)$ for $\cos(x)$ at $a=0$

$$T_n(x) = 0 + 1\cdot x + \frac{0\cdot x^2}{2!} + \frac{(-1)}{3!}x^3 + \frac{0\cdot x^4}{4!} + \frac{x^5}{5!} + \frac{0\cdot x^6}{6!} - \frac{1\cdot x^7}{7!} \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Example $y = \ln(x)$ at $x = 1$.



$f(x) = \ln(x)$ $\quad f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\quad f'(1) = 1$

$f''(x) = -x^{-2}$ $\quad f''(1) = -1$

$f^{(3)}(x) = 2x^{-3}$ $\quad f^{(3)}(1) = 2$

$f^{(4)}(x) = 2\cdot(-3)x^{-4}$ $\quad f^{(4)}(1) = -2\cdot 3$

$f^{(5)}(x) = 2\cdot(-3)(-4)x^{-5}$ $\quad f^{(5)}(1) = 2\cdot 3\cdot 4 = (5-1)!$

$f^{(k)}(x) = (-1)^{k+1}(k-1)!\,x^{-k}$ $\quad f^{(k)}(1) = (-1)^{k+1}(k-1)!$

so $T_n(x)$ for $\ln(x)$ at $x = 1$

$$T_n(x) = 0 + 1(x-1) + \frac{(-1)}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{2\cdot 3}{4!}(x-1)^4 + \dots + \frac{(-1)^{k+1}(k-1)!}{k!}(x-1)^k$$

$$= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

Q: how good is the approximation?

Thm (Error bound) $f(x)$ function $f^{(n+1)}(x)$ exists and is continuous.

let K be an upper bound on $|f^{(n+1)}(u)|$ for all u in $[a,x]$.

then $$|T_n(x) - f(x)| \leq \frac{K\,|x-a|^{n+1}}{(n+1)!}$$

Example $f(x) = e^x$ find $T_4(x)$ at $x=0$ find an error bound for $T(1)$.

$f(x) = e^x \quad f(0) = 1$

$f'(x) = e^x \quad f'(0) = 1 \qquad T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$f''(x) = e^x \quad f''(0) = 1 \qquad T_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \approx .$

error bound: need to find K s.t. $|f^{(n+1)}(u)| \leq K$ for $u \in [0,1]$.

i.e. $|e^u| \leq K$ for $u \in [0,1]$; can choose $K = e$

so error $\leq \frac{e \cdot 1^{n+1}}{(n+1)!} = \frac{e \cdot 1^5}{5!} = \frac{e}{120}$.

Example $f(x) = \frac{1}{1-x}$

$f(x) = \frac{1}{1-x} = (1-x)^{-1} \qquad f(0) = 1$

$f'(x) = +(1-x)^{-2} \qquad f'(0) = 1$

$f''(x) = (-1)(-2)(1-x)^{-3} \qquad f''(0) = 2!$

$f^{(3)}(x) = (-1)(-2)(-3)(1-x)^{-4} \qquad f^{(3)}(0) = 3!$

$f^{(k)}(x) = k!\,(1-x)^{-k+1} \qquad f^{(k)}(0) = k!$

$$T_n(x) = 1 + 1\cdot x + \frac{2!\,x^2}{2!} + \frac{3!\,x^3}{3!} + \cdots = 1 + x + x^2 + x^3 + x^4 + \cdots$$

§10.1 Sequences

A sequence is a list of numbers, indexed by $\mathbb{N}$ = positive integers.

Examples $1, 2, 3, 4, 5, \ldots$ Notation $\{a_n\}_{n \in \mathbb{N}}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ a_n is the n-th number

$1, 27, \pi, \sqrt{2}, \ldots$

$\{a_n\}_{n \in \mathbb{N}} \leftrightarrow a_1, a_2, a_3, a_4, \ldots$

Sometimes we can give the sequence by a formula (depending on n)

Examples $\{a_n\} = \{n\} = 1, 2, 3, 4, \ldots$

$\{a_n\} = \{\frac{1}{n}\} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

But we don't need to.

Example (recursive def^{n}) $a_{n+2} = a_{n+1} + a_n$. $a_1 = 1$ $a_2 = 1$.

gives $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ (Fibonacci sequence).

Sequences may converge.

Def^{n} A sequence $\{a_n\}$ converges to L if for every $\epsilon > 0$ there is an N s.t. $|a_n - L| \leq \epsilon$ for all $n \geq N$.

Notation $\lim_{n \to \infty} a_n = L$ or $a_n \to L$

Examples $\{a_n\} = \{\frac{1}{n}\}$ $a_n = \frac{1}{n}$. $\lim_{n \to \infty} \frac{1}{n} = 0$.

Check pick $\epsilon > 0$ then choose $N > \frac{1}{\epsilon}$.

Special case sequence defined by a function $f(x)$, i.e. $a_n = f(n)$

Thm If $\lim_{x \to \infty} f(x) = L$, then $\lim_{n \to \infty} f(n) = L$.