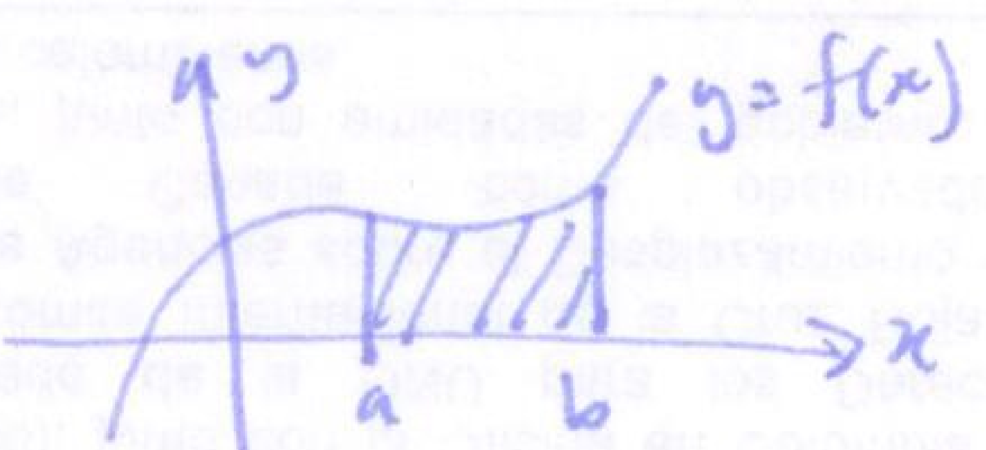
note $\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + c$ $\quad \int \frac{x}{1+x^2}dx = \frac{1}{2}\ln|1+x^2| + c$

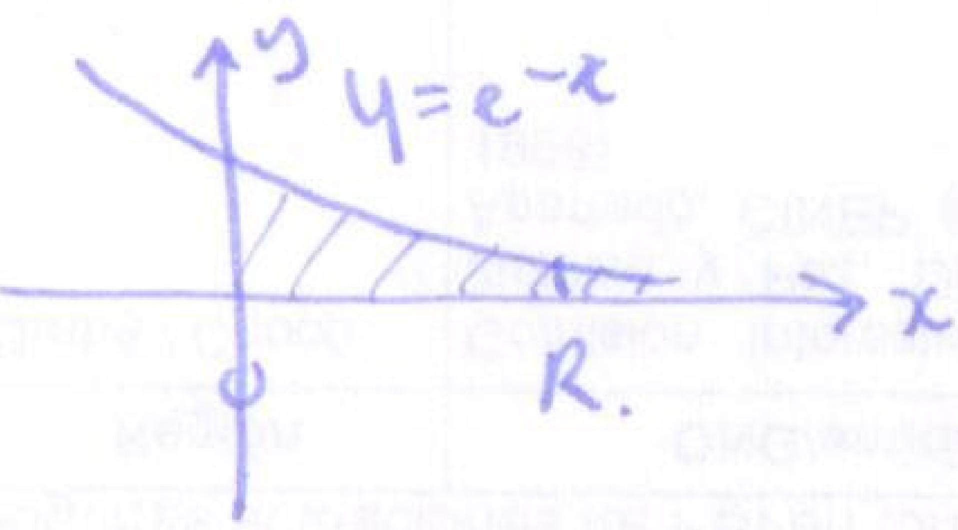
7.7 Improper integrals

recall $\int_a^b f(x)dx$ = area under the curve



Q: what about infinite intervals?

Example.



$\int_0^\infty e^{-x}dx$ note $\int_0^R e^{-x}dx = \left[-e^{-x}\right]_0^R.$

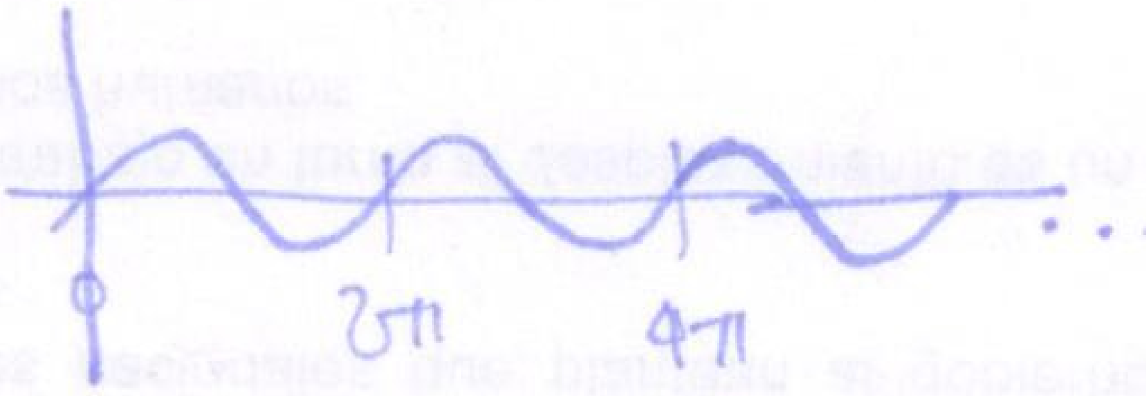
$= -e^{-R} + e^{0} = 1 - e^{-R}.$

Def^n $\int_a^\infty f(x)dx = \lim_{R\to\infty}\int_a^R f(x)dx$ if this limit exists.

Example $\lim_{R\to\infty}\int_0^R e^{-x}dx = \lim_{R\to\infty} 1 - e^{-R} = 1$

Warning: sometimes the limit doesn't exist.

Example: $\int_0^\infty \sin(x)dx$



$\int_0^R \sin(x)dx = \left[\cos(x)\right]_0^R.$

$= -\cos(R) - (-\cos(0)).$

$= 1 - \cos(R)$

$\lim_{R\to\infty}\int_0^R \sin(x)dx$ DNE. $\quad \lim_{N\to\infty}\int_0^{2\pi N}\sin(x)dx = 0$

$\lim_{N\to\infty}\int_0^{2\pi N+\pi}\sin(x)dx = 2$!

Example $\int_1^\infty \frac{1}{x^2}dx$

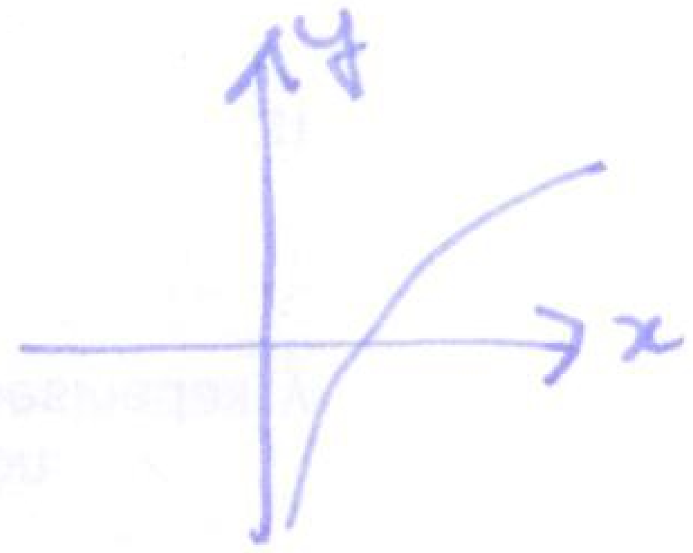
$\lim_{R\to\infty}\int_1^R \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^R.$

$= -\frac{1}{R} + 1$

$\lim_{R\to\infty} 1 - \frac{1}{R} = 1$

Example $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{x}\,dx = \lim_{R\to\infty}\Big[\ln|x|\Big]_1^R$

$$= \lim_{R\to\infty} \ln R - 0 = \infty.$$



Doubly infinite integrals $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$

Defn

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx \leftarrow \text{provided each one exists!}$$

Example

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_{0}^{\infty}\frac{1}{1+x^2}\,dx$$

$$= \lim_{R\to\infty}\int_{-R}^{0}\frac{1}{1+x^2}\,dx + \lim_{R\to\infty}\int_{0}^{R}\frac{1}{1+x^2}\,dx$$

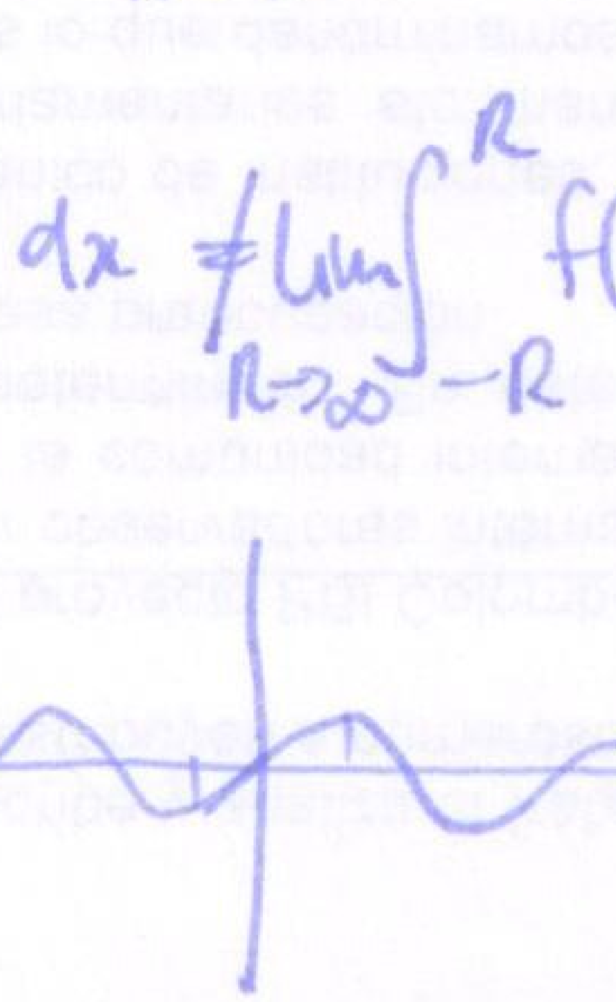
$$= \lim_{R\to\infty}\Big[\tan^{-1}(x)\Big]_{-R}^{0} + \lim_{R\to\infty}\Big[\tan^{-1}(x)\Big]_{0}^{R}.$$

$$= \lim_{R\to\infty}\left(0 - \tan^{-1}(-R)\right) + \lim_{R\to\infty}\left(\tan^{-1}(R) - 0\right).$$

$$= \lim_{R\to\infty}\tan^{-1}(R) + \lim_{R\to\infty}\tan^{-1}(R) = \pi.$$

Warning $\displaystyle\lim \int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx.$

Example $\displaystyle\int_{-\infty}^{\infty}\sin(x)\,dx$ $\qquad \displaystyle\lim_{R\to\infty}\int_{-R}^{R}\sin(x)\,dx = 0$



but $\displaystyle\lim_{R\to\infty}\int_{0}^{R}\sin(x)\,dx$ DNE.