§7.6 Partial fractions

aim: $\int \frac{P(x)}{Q(x)}\, dx$ P, Q polynomials.

note: $\int \frac{1}{x+a}\, dx = \ln|x+a| + c$

all a_i distinct!

claim if $\deg P < \deg Q$ and $Q(x) = (x+a_1)(x+a_2)\ldots(x+a_n)$

then $\frac{P(x)}{Q(x)} = \frac{A_1}{x+a_1} + \frac{A_2}{x+a_2} + \ldots + \frac{A_n}{x+a_n}$ A_i numbers.

Example $\int \frac{x}{x^2-1}\, dx$ $x^2-1 = (x+1)(x-1)$

$$\frac{x}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}.$$

$$= \frac{x(A+B)+(B-A)}{x^2-1}$$

$$x = 1.x + 0 = x\underbrace{(A+B)}_{1} + \underbrace{(B-A)}_{0}.$$

$A+B = 1$ ①
$B-A = 0$ ②

①+②: $2B = 1$
$B = 1/2$
$A = 1/2$.

$$\int \frac{x}{x^2-1}\, dx = \int \frac{1/2}{x+1} + \frac{1/2}{x-1}\, dx = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + c.$$

Example spose $\deg P \geq \deg Q$? Long division

$$\frac{x^3}{x^2-1} = \begin{array}{r|l} & x \\ \hline x^2-1 & x^3 \\ & x^3 - x \\ \hline & x \end{array}$$

so $\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$.

$$\int \frac{x^3}{x^2-1}\, dx = \int x + \frac{x}{x^2-1}\, dx = \tfrac{1}{2}x^2 + \underline{\qquad} + c.$$

repeated factors: $\displaystyle\int \frac{x+1}{(x-1)(x+2)^2}\,dx$

try $$\frac{x+1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

$$= \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}.$$

$$= \frac{A(x^2+4x+4) + B(x^2+x-2) + C(x-1)}{(x-1)(x+2)^2}.$$

$$= \frac{x^2\overset{0}{(A+B)} + x\overset{1}{(4A+B+C)} + \overset{1}{4A-2B-C}}{(x-1)(x+2)^2}.$$

$B=-A$

$$\left.\begin{aligned} A+B &= 0 \\ 4A+B+C &= 1 \\ 4A-2B-C &= 1 \end{aligned}\right\} \qquad \left.\begin{aligned} 3A+C &= 1 \quad (1) \\ 6A-C &= 1 \quad (2) \end{aligned}\right\} \quad (1)+(2) \quad 9A = 2 \quad A = \frac{2}{9}.$$

$$C = 6A+1 = \frac{4}{3}+1 = \frac{7}{3}.$$

$$B = -\frac{2}{9}.$$

$$\int \frac{2/9}{x-1} + \frac{-2/9}{x+2} + \frac{7/3}{(x+2)^2}\,dx$$

$$= \frac{2}{9}\ln|x-1| - \frac{2}{9}\ln|x+2| + \frac{7}{3}\,\frac{1}{x+2} + c$$

Quadratic factors

$x^2+1 \neq (x+a_1)(x+a_2)$ for a_1, a_2 real.

trick: $$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)+c$$

Q: $$\int \frac{1}{1+x+x^2}\,dx$$ complete the square: $x^2+x+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$.

$$= \int \frac{1}{3/4+\left(x+\frac{1}{2}\right)^2}\,dx$$ sub $u = x+\frac{1}{2}$, $\frac{du}{dx}=1$ $$\int \frac{1}{3/4+u^2}\,du.$$ etc...

~~Repeated quadratic factors~~ linear and quadratic factors

$$\int \frac{1}{x(1+x^2)}\,dx \qquad \frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + (Bx+C)x}{x(x^2+1)}$$

$$= \frac{x^2\overbrace{(A+B)}^{0} + x\overbrace{(C)}^{0} + \overbrace{A}^{1}}{x(x^2+1)}$$

$A = 1$
$B = -1$
$C = 0$

$$= \int \frac{1}{x} + \frac{-x}{x^2+1}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + c$$

Repeated ~~two~~ quadratic factors.

$$\frac{1}{(x+a)(x+b)^2(x^2+c)^2} = \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{(x+b)^2} + \frac{Dx+E}{(x^2+c)} + \frac{Fx+G}{(x^2+c)^2}.$$

solve for $A, B, \ldots G$.