§7.3 Trig integrals $\int \sin^m x \cos^n x \, dx$?

tricks
- $\cos^2 x + \sin^2 x = 1$
- $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$.
- sub $u = \sin x$, $\frac{du}{dx} = \cos x$
- sub $u = \cos x$, $\frac{du}{dx} = -\sin x$
- parts.

Examples $\int \sin^2 x \, dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx. = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$.

$\int \sin^3 x \, dx = \int (1-\cos^2 x)\sin x \, dx$ sub $u = \cos x$, $\frac{du}{dx} = -\sin x$

$= \int (1-u^2)\sin x \frac{du}{-\sin x} = -\int 1 - u^2 \, du$.

moral: if there is an odd power want to do sub $u =$ ~~odd power trig function~~ other trig function.

$\int \sin^4 x \underbrace{\cos^3 x}_{\text{odd power.}} dx$ ~~want~~ try: $u = \sin x$, $\frac{du}{dx} = \cos x$.

$= \int \sin^4 x \cos^2 x \cdot \cos x \, dx = \int \sin^4 x (1 - \sin^2 x) \cos x \, dx$

$= \int u^4 (1-u^2) \cos x \frac{du}{\cos x} = \int u^4 (1-u^2) \, du$.

even powers $\int \sin^4 x \cos^2 x \, dx$? get everything in term of either $\sin(x)$ or $\cos(x)$, then use parts.

$= \int \sin^4 x (1 - \sin^2 x) \, dx$

$= \int \sin^4 x - \sin^6 x \, dx$.

$$\int \sin^6 x\, dx = \int \underset{u}{\sin^5 x} \cdot \underset{v'}{\sin x}\, dx \qquad \int uv'\, dx = uv - \int u'v\, dx$$ (19)

$$u = \sin^5 x \quad u' = 5\sin^4 x \cos x$$
$$v' = \sin x \quad v = -\cos x.$$

$$\int \sin^6 x\, dx = -\sin^5 x \cos x - \int 5\sin^4 x \cos x \cdot (-\cos x)\, dx$$

$$\int \sin^6 x\, dx = -\sin^5 x \cos x + 5\int \sin^4 x \cos^2 x\, dx$$

$$\int \sin^6 x\, dx = -\sin^5 x \cos x + 5\int \sin^4 x\,(1 - \sin^2 x)\, dx$$

$$\int \sin^6 x\, dx = -\sin^5 x \cos x + 5\int \sin^4 x\, dx - 5\int \sin^6 x\, dx$$

$$6\int \sin^6 x\, dx = -\sin^5 x \cos x + \underbrace{5\int \sin^4 x\, dx}_{\text{do by parts!}}$$

other trig functions

recall: $\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}\,dx$ $\quad u = \cos x \quad \frac{du}{dx} = -\sin x$ $\quad = \int \frac{\sin x}{u} \cdot \frac{-1}{\sin x}\,du = -\int \frac{1}{u}\,du$

$= -\ln|u| + c = -\ln|\cos(x)| + c = \ln|\sec(x)| + c.$

~~ex~~ fact: $\int \sec(x)\,dx = \ln|\sec x + \tan x| + c$ (check by differentiating).

$\int \tan^m x \sec^n x\,dx$

tricks

$\cos^2 x + \sin^2 x = 1.$

- $1 + \tan^2 x = \sec^2 x.$
- $u = \sec x \quad \frac{du}{dx} = \sec x \tan x$
- $u = \tan x \quad \frac{du}{dx} = \sec^2 x.$
- parts.

m odd: $\int \tan^3 x \sec^2 x\,dx = \int \tan^2 x \sec x\,(\tan x \sec x)\,dx.$

$= \int (1 - \sec^2 x) \sec x\,(\tan x \sec x)\,dx$ $\quad u = \sec x \quad \frac{du}{dx} = \sec x \tan x$

$= \int (1-u^2)\,u \frac{\tan x \cancel{\sec x}}{\tan x \cancel{\sec x}}\,du.$

n even: $\int \tan^3 x (\sec^2 x)\,dx$ $\quad u = \tan x \quad \frac{du}{dx} = \sec^3 x.$ $\quad = \int u^3 \frac{\sec^2 x}{\sec^2 x}\,du = \int u^3\,du.$

m even, n odd: write as powers of sec(x) and use integration by parts.

$\int \sin 3x \cos 2x \, dx$? useful fact: $\sin(A+B) = \cancel{\sin A \sin B + \cos}$

$\sin A \cos B + \cos A \sin B$.

$\cos(A+B) = \cos A \cos B - \sin A \sin B$.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

$$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$= \sin A \cos(B) - \cos A \sin B.$$

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B.$$

$$\int \sin 3x \cos 2x \, dx = \frac{1}{2} \int \sin 5x + \sin x \, dx.$$

$\int \cos 4x \cos 7x \, dx$? $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\cos(A-B) = \cos A \cos(-B) - \sin A \sin(-B)$$

$$= \cos A \cos B + \sin A \sin B.$$

$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B.$$

$$\int \cos 4x \cos 7x \, dx = \frac{1}{2} \int \cos 11x + \cos(-3x) \, dx.$$

§7.4 Trig substitution

aim: $\int \sqrt{a^2-x^2}\,dx$ ① $\sqrt{a^2+x^2}$ ② $\sqrt{x^2-a^2}$ ③

trick: use: $\cos^2 x + \sin^2 x = 1 \iff$ ① $\boxed{\cos^2 x = 1 - \sin^2 x}$, $\sin^2 x = 1 - \cos^2 x$. to convert $\sqrt{\ }$ into perfect square.

$\boxed{1 + \tan^2 x = \sec^2 x}$ ② $\iff \boxed{\tan^2 x = \sec^2 x - 1}$ ③.

Example.

- $\int \sqrt{9-x^2}\,dx$ try $x = 3\sin u$.

  $\frac{dx}{du} = 3\cos u$.

  $\int \sqrt{9-(3\sin u)^2}\,\frac{dx}{du}\,du$

  $\int \sqrt{9 - 9\sin^2 x}\; 3\cos u\,du = 3\int \sqrt{\underbrace{1-\sin^2 x}_{=\cos^2 x}}\; 3\cos u\,du$.

  $= 9\int \cos^2 u\,du$ ... etc.

- $\int \sqrt{1+4x^2}\,dx$ try $x = \frac{1}{2}\tan u$.

  $\frac{dx}{du} = \frac{1}{2}\cancel{\tan u}\sec^2 u$.

  $\int \sqrt{1 + 4\cdot\frac{1}{4}\tan^2 u}\;\frac{dx}{du}\,du = \int \sqrt{\underbrace{1+\tan^2 u}_{\sec^2 u}}\;\frac{1}{2}\sec^2 u\,du$.

  $= \frac{1}{2}\int \sec^3 u\,du = \frac{1}{2}\int \sec^3 u\,du$. ~~try~~ $v = \tan u$.

  as before: parts or use $\sec^2 u = 1 + \tan^2 u$.

- $\int \frac{1}{x^2\sqrt{x^2-a^2}}\,dx$ try $x = a\sec u$

  $\frac{dx}{du} = a\sec u\tan u$

  $\int \frac{1}{a^2\sec^2 u}\,\frac{1}{\sqrt{a^2\sec^2 u - a^2}}\,\frac{dx}{du}\,du = \int \frac{1}{a^2\sec^2 u}\,\frac{1}{a\tan u}\cdot a\sec u\tan u\,du$

  $= \frac{1}{a^2}\int \cos u\,du = \frac{1}{a^2}\sin u + c$ get back in terms of x .... $\sin u = \frac{\sqrt{x^2-a^2}}{x}$.

  $x = a\sec u$

  $\frac{x}{a} = \sec u$ $\frac{a}{x} = \cos u$.