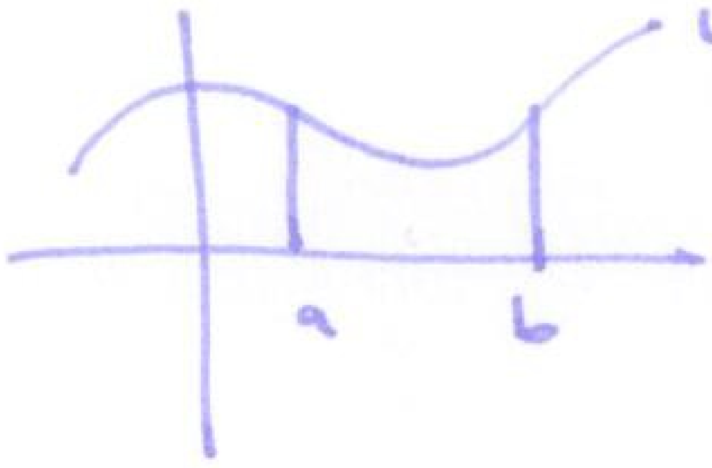
Error bounds



Let K_2 be an upper bound for $f''(x)$, i.e.

$$|f''(x)| \le K_2 \quad \text{for all } x \in [a,b]$$

then

$$\text{Error}(T_N) \le \frac{K_2(b-a)^3}{12N^2} \qquad \text{Error}(M_N) \le \frac{K_2(b-a)^3}{24N^2}.$$

Example You want to compute $\int_0^1 e^{-x^2}\,dx$ to 4 decimal places.

$f(x) = e^{-x^2}$. $f'(x) = e^{-x^2}\cdot(-2x)$ $f''(x) = e^{-x^2}(-2x)^2 + e^{-x^2}\cdot(-2)$

$= -e^{-x^2}(4x^2+2)$

for $|f''(x)|$.

find upper bound on $[0,1]$: $\left.\begin{array}{l}|e^{-x^2}| \le 1 \\ |4x^2+2| \le 6.\end{array}\right\}$ so $|f''(x)| \le 6$

so $\text{Error}(T_N) \le \dfrac{6}{12N^2} = 10^{-4}$ $\quad N^2 = \dfrac{1}{2}\cdot 10^{-5}$ $\quad N \approx \sqrt{\tfrac{1}{2}\cdot 10^{+5}}$?

Observation

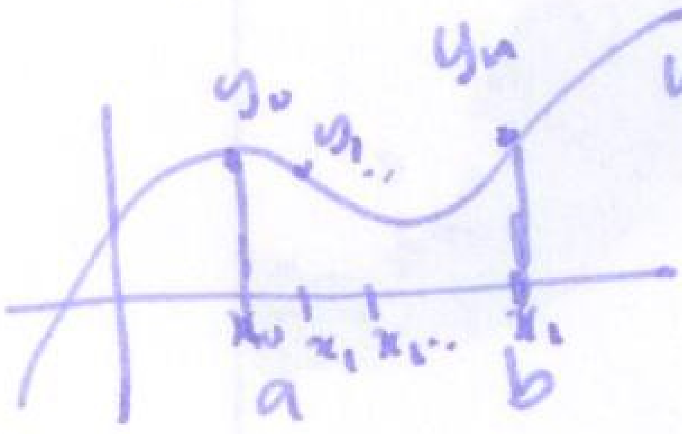
$T_N < \text{area} \le M_N$

Simpsons rule

$$S_N = \frac{1}{3}T_{N/2} + \frac{2}{3}M_{N/2}.$$

$\Delta x = \dfrac{b-a}{N}$ why: quadratic approx.

formula.



$$S_N = \frac{1}{3}\Delta x\left(y_0 + 4y_1 + 2y_2 + \cdots + 4y_{N-3} + 2y_{N-2} + 4y_{N-1} + y_N\right)$$

$$\text{Error}(S_N) \le \frac{K_4(b-a)^5}{180N^4} \qquad |f^{(4)}(x)| \le K_4$$