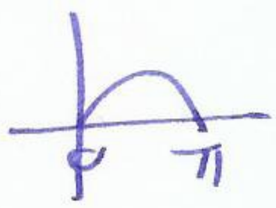
so $\int_a^b f(x)\,dx \approx \underbrace{\left(f(x_1)+f(x_2)+\cdots+f(x_n)\right)\frac{1}{n}}_{\text{average}}(b-a)$

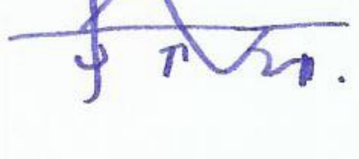
so average $= \frac{1}{b-a}\int_a^b f(x)\,dx.$

Example Find average value of $\sin(x)$ on $[0,\pi]$, $[0,2\pi]$.

$[0,\pi]$: $\frac{1}{\pi}\int_0^{\pi}\sin(x)\,dx = \frac{1}{\pi}\left[-\cos(x)\right]_0^{\pi} = \frac{1}{\pi}\left(-(-1)+1\right) = \frac{2}{\pi}.$

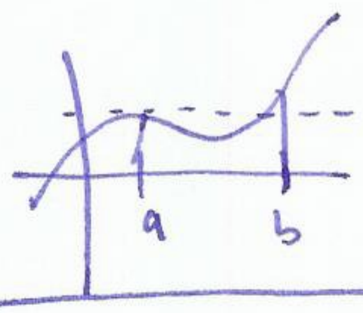


$[0,2\pi]$: $\frac{1}{2\pi}\int_0^{2\pi}\sin(x)\,dx = \frac{1}{2\pi}\left[-\cos(x)\right]_0^{2\pi} = \frac{1}{2\pi}\left(-1+1\right) = 0$

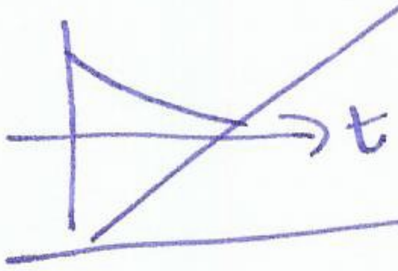


Useful fact (Mean value theorem for integrals)

If $f(x)$ is continuous on $[a,b]$ then there is a $c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$



Example Bus waiting times are exponentially distributed as $\lambda e^{-\lambda t}$



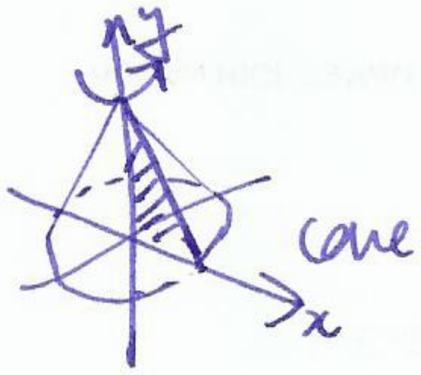
Q. what is the average waiting time?

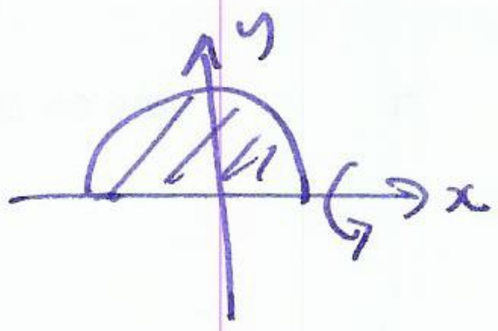
Example find the average value of $\frac{\sin(\pi/x)}{x^2}$ over $[1,2]$.

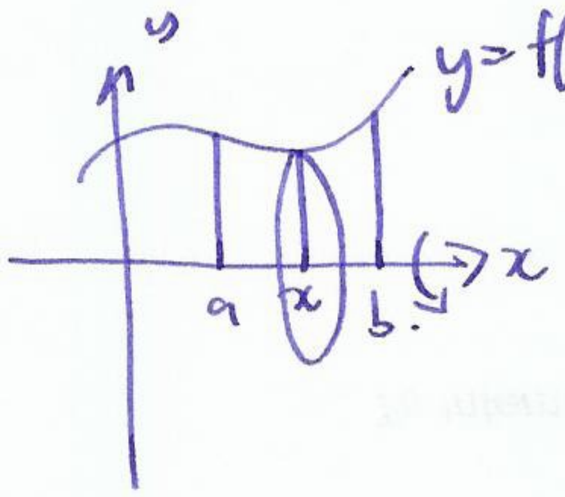
Low cunning: which has the bigger average on $[0,\pi]$ $\sin(x)$ or $\sin^2(x)$?

§6.3 Volumes of revolution

Examples.

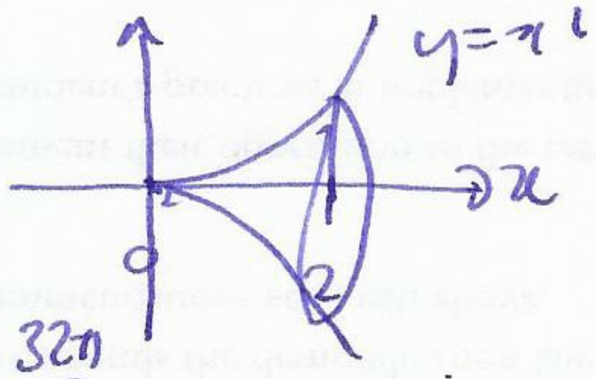






find vertical cross section $A = \pi (f(x))^2$.

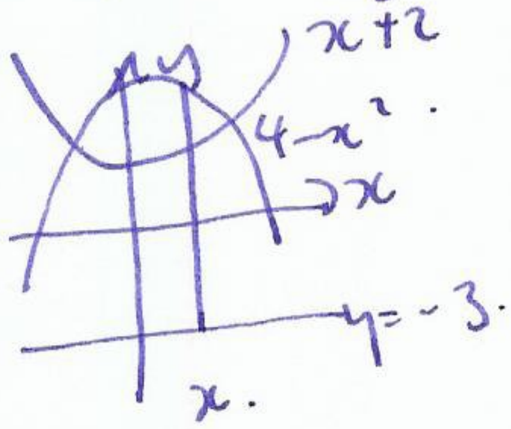
volume $V = \int_a^b \pi (f(x))^2 \, dx$



Example rotate $y = x^2$ about x-axis.

$$\pi \int_0^2 (x^2)^2 \, dx = \pi \left[\tfrac{1}{5} x^5 \right]_0^2 = \frac{\pi 2^5}{5} = \frac{32\pi}{5}.$$

Example rotate area between $f(x) = x^2 + 2$ and $g(x) = 4 - x^2$ about $y = -3$.



intersections: $x^2 + 2 = 4 - x^2$

$2x^2 - 4 + 2 = 0$

~~$2(x^2 - 2 + 1) = 0$~~ $(x-1)^2$

$x^2 = \pm 1$.

$y = -3$.

$\pi \left((g(x)+3)^2 - (f(x)+3)^2 \right)$.

Cross-sectional area: x-axis.

$\pi (g(x))^2 - \pi (f(x))^2$

$\pi \int_{-1}^{1} (4-x^2)^2 - (x^2+2)^2 \, dx$

$\pi \int_{-1}^{1} (7-x^2)^2 - (x^2+5)^2 \, dx = 32\pi$.