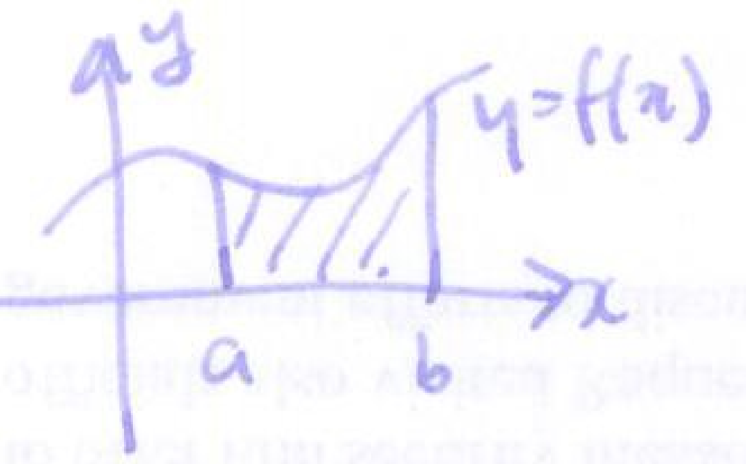
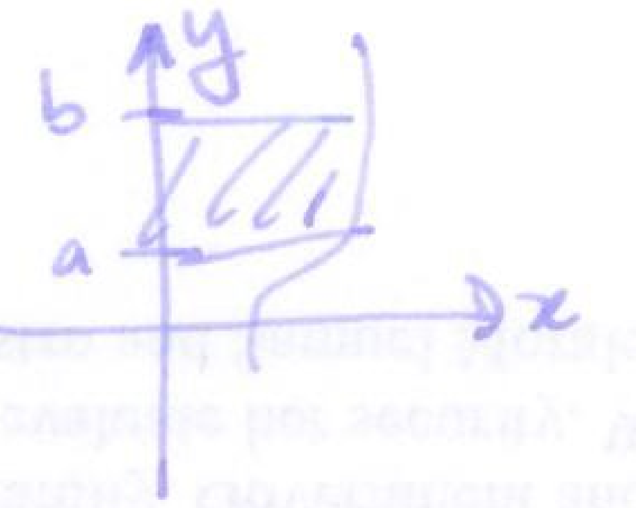
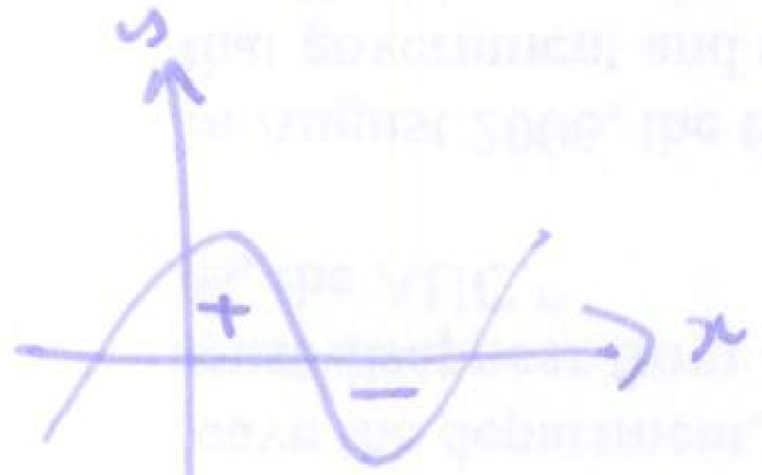
# Integration along the y-axis

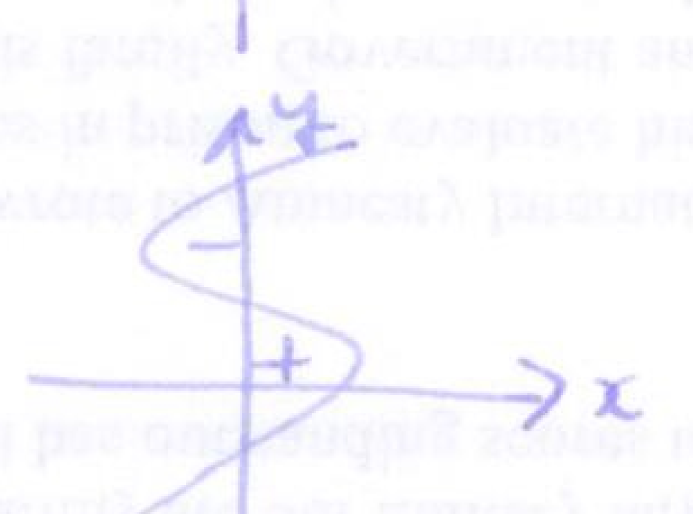
$\int_a^b f(x)\,dx$



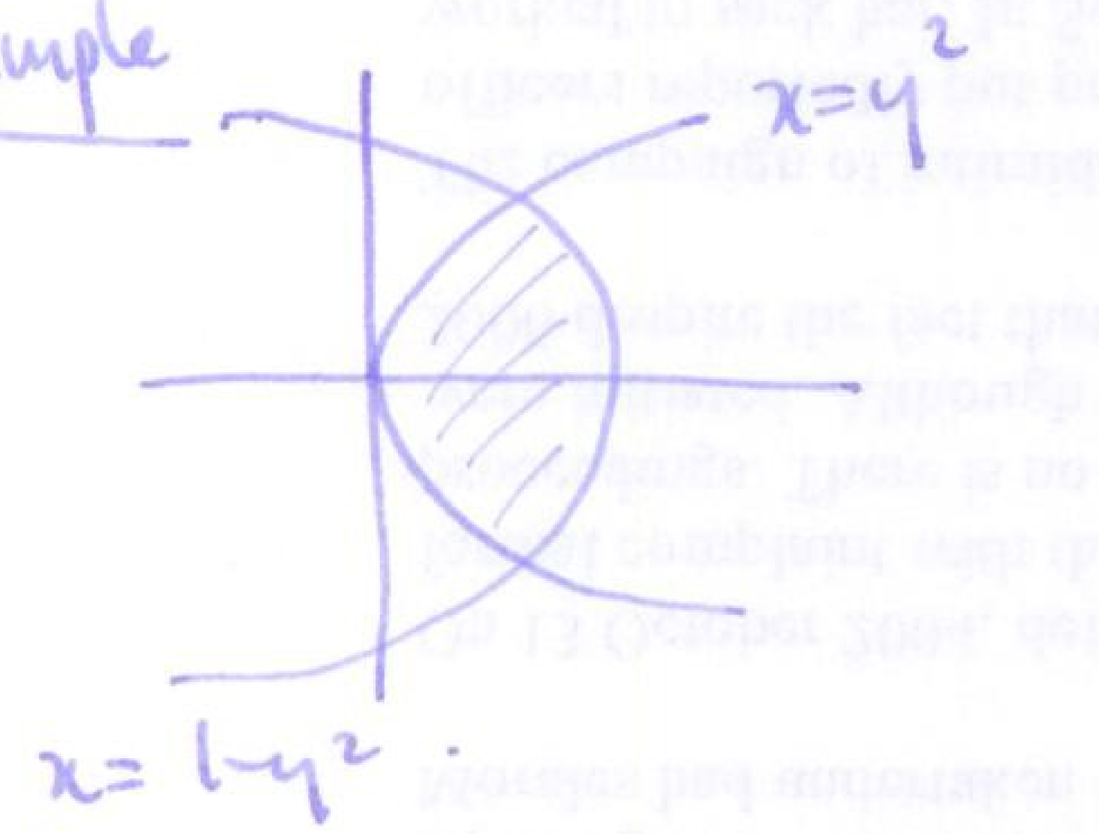

$\int_a^b f(y)\,dy$




Example




find this area.

## §6.2 Volume, density, averages

Volume: recall volume of cylinder is

$Ah$ $\quad A =$ area of base.

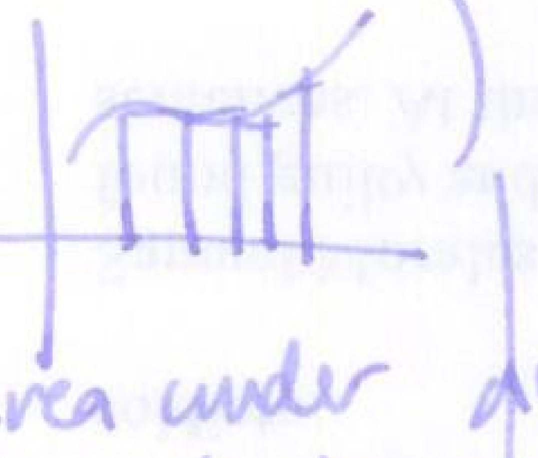
this works for cylinders of any shape

$Ah$.

Suppose the shape is not a cylinder

can approximate volume by horizontal slices.

(recall:




approx area under curves by rectangles of width $\Delta x$)

approx volume of object by cylinders.

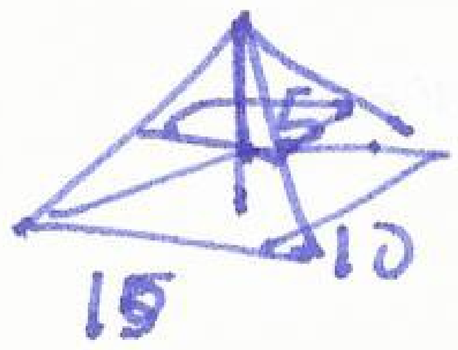
$= $ cross section $\times$ height (height $\Delta y$ say)

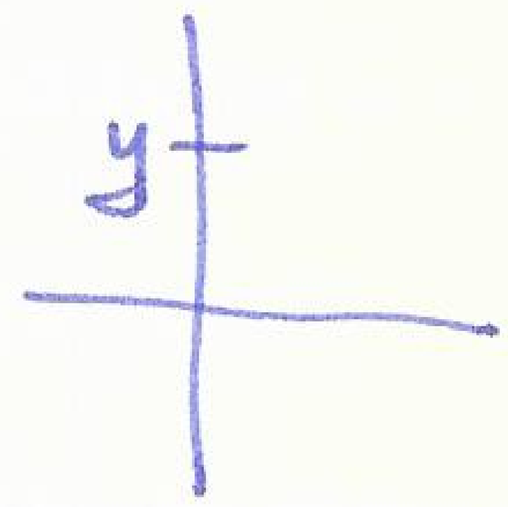
volume $V \simeq \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} A(y_i)\Delta y$

where $A(y)$ = area of horizontal cross section at height $y$

$$\boxed{\text{Volume } V = \int_a^b A(y)\,dy}$$

<u>Example</u> find volume of square pyramid : base 10m x 10m height 5m




at height $y$ side length is $\cancel{5 - \frac{y}{2}}$.

$\cancel{5 - 2y}$ $\quad 10 - 2y$

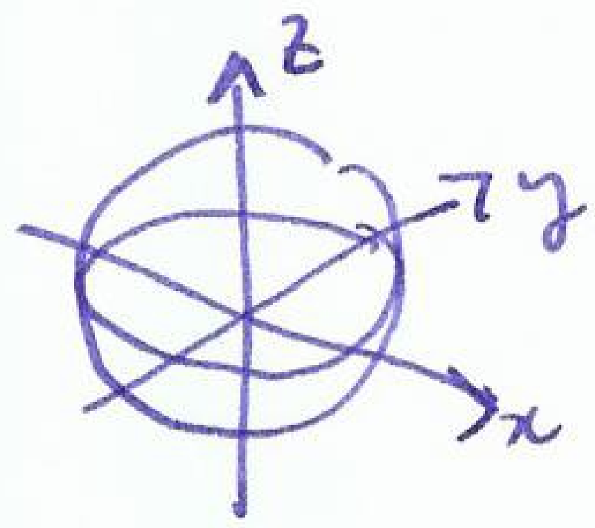
area is $(10-2y)^2$

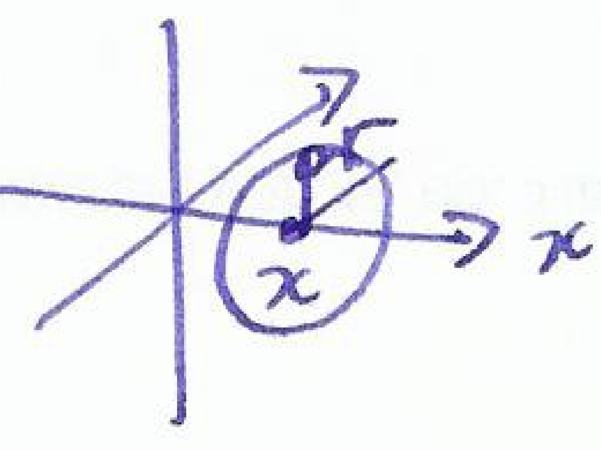
so volume is $\displaystyle\int_0^5 (10-2y)^2 \, dy = \int_0^5 100 - 40y + 4y^2 \, dy$

$= \left[ 100y - 20y^2 + \frac{4}{3}y^3 \right]_0^5 = 500 - 20 \cdot 25 + \frac{4}{3} 125 = \cancel{4}\ \frac{500}{3}.$

$25.20$

**Example** volume of sphere of radius $R$




area of cross section radius at $x$ is $r = \sqrt{R^2 - x^2}$.

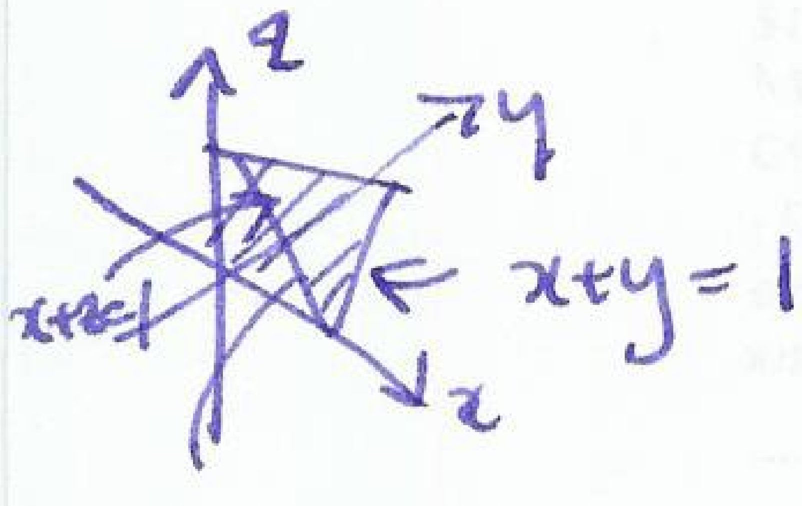
$x^2 + r^2 = R^2$ so area $A = \pi r^2 = \pi(R^2 - x^2)$

so volume is $\int_{-R}^{R} \pi(R^2 - x^2)\,dx = \pi\left[R^2x - \frac{1}{3}x^3\right]_{-R}^{R} = \pi\left(R^3 - (-R)^3 - \frac{1}{3}R^3 + \frac{1}{3}(-R)^3\right)$

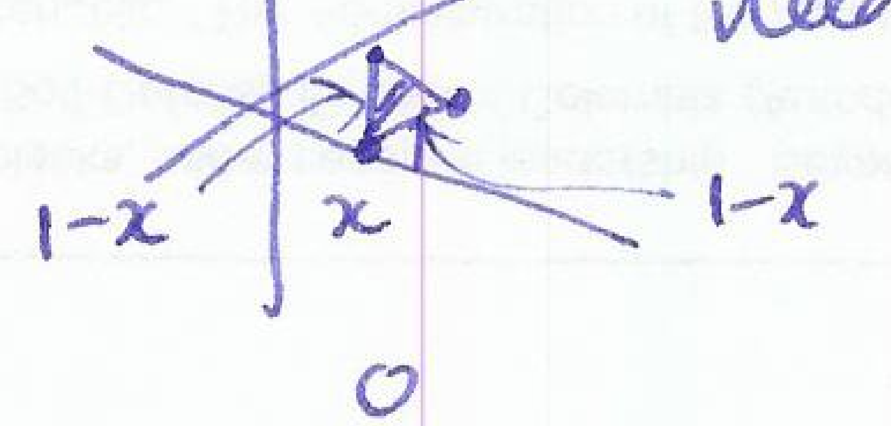
$= \pi R^3\left(2 - \frac{2}{3}\right) = \frac{4}{3}\pi R^3.$

**Example** volume of tetrahedron with vertices

$(0,0,0)$ $(1,0,0)$ $(0,1,0)$ $(0,0,1)$ (or volume in +ve octant below $x+y+z=1$)




cross section: need area of this triangle.




$A = \frac{1}{2}bh = \frac{1}{2}(1-x)^2.$

$x+y+z=1$

$$vol = \int_0^1 \frac{1}{2}(1-x)^2\,dx = \int_1^0 \frac{1}{2}u^2(-1)\,du = \left[-\frac{1}{6}u^3\right]_1^0 = \frac{1}{6}.$$

$u = 1-x$

$\frac{du}{dx} = -1$

**Average value** of numbers $a_1, a_2, \ldots a_n = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i$

Q: what about average value of function $f(x)$ on an interval?


average $= \cancel{(f(x_1) + f(x_2) + \cdots + f(x_n))}$.

$= \frac{1}{n}\left(f(x_1)\cancel{\Delta x} + f(x_2)\cancel{\Delta x} + \cdots + f(x_n)\cancel{\Delta x}\right).$

$= \left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)\frac{\cancel{\Delta x}1}{n}$

$= \left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)\cancel{\frac{(b-a)}{n}}$

recall $\int_a^b f(x)\,dx \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$ $\quad \Delta x = \frac{b-a}{n}$