§5.6 Integration by substitution

$$\int f(x)\,dx \quad , \quad x(u) \quad \rightsquigarrow \quad \int f(x(u))\,\frac{dx}{du}\,du$$

integration by substitution is a "reverse chain rule". recall $\frac{dx}{du} = 1/\frac{du}{dx}$

Example · $\int x\sin(x^2)\,dx$ try $x^2 = u$: $\frac{du}{dx} = 2x$. [look for chain rule].

$$\rightsquigarrow \int x\sin(u)\frac{dx}{du}\,du = \int x\sin(u)\frac{1}{2x}\,du = \int \sin(u)\,du$$

$$= -\cos(u) + c = -\cos(x^2) + c. \qquad \text{check!}$$

· $\int e^{4x}\,dx$, $u = 4x$, $\frac{du}{dx} = 4$: $\int e^u \cdot \frac{1}{4}\,du = \frac{1}{4}e^u + c = \frac{1}{4}e^{4x} + c$. check!

· $\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta$ [look for: $\frac{d}{dx}(\ln(f(x))) = \frac{1}{f(x)}\cdot f'(x)$].

try $u = \cos\theta$, $\frac{du}{d\theta} = -\sin\theta$

$$\int \frac{\sin\theta}{u}\cdot\frac{1}{-\sin\theta}\,du = -\int \frac{1}{u}\,du = -\ln|u| + c$$
$$= -\ln|\cos x| + c$$

check!

· $\int x\sqrt{x+1}\,dx$ try: $u = x+1$, $\frac{du}{dx} = 1$ $\quad \int x\sqrt{u}\,du = \int (u-1)\sqrt{u}\,du$

$$= \int u^{3/2} - u^{1/2}\,du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + c.$$

Definite integrals

Example $\int_0^2 x^2\sqrt{x^3+1}\,dx$.

$$\int_a^b f(x)\,dx \;,\; x(u) \quad \rightsquigarrow \quad \int_{x^{-1}(a)}^{x^{-1}(b)} f(x(u))\,\frac{dx}{du}\,du$$

$$\int_{x=a}^{x=b} \quad \rightsquigarrow \quad \int_{u=?}^{u=?}$$

§5.7 More functions

inverse trig functions

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + c$$

$$\cancel{\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}} \qquad \cancel{\int \tan^{-1}(x)\,dx}$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + c$$

$$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + c$$

Examples $\int_0^{\cancel{2}1} \frac{1}{1+x^2}\,dx = \left[\tan^{-1}(x)\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

$\int_0^1 \frac{1}{4+x^2}\,dx$. sub $x = 2u$, $x^2 = 4u^2$, $\frac{dx}{du} = 2$. $\rightsquigarrow \int_0^{1/2} \frac{1}{4+4u^2}\,2\,du$

$= \frac{1}{\cancel{4}2}\int_0^{1/2} \frac{1}{1+u^2}\,du = \frac{1}{2}\left[\tan^{-1}(u)\right]_0^{1/2} = \frac{1}{2}\left(\tan^{-1}(1/2) - \tan^{-1}(0)\right) = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$

$\int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}\,dx$ try $u = \cancel{3x}\,x/3$, $\frac{du}{dx} = \cancel{3}1/3$ $\qquad \int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\,\frac{1}{3}\,du = \left[\frac{1}{3}\tan^{-1}(u)\right]_0^{1/2}$.

exponential functions

$$\frac{d}{dx}(e^x) = e^x \qquad \int e^x\,dx = e^x + c$$

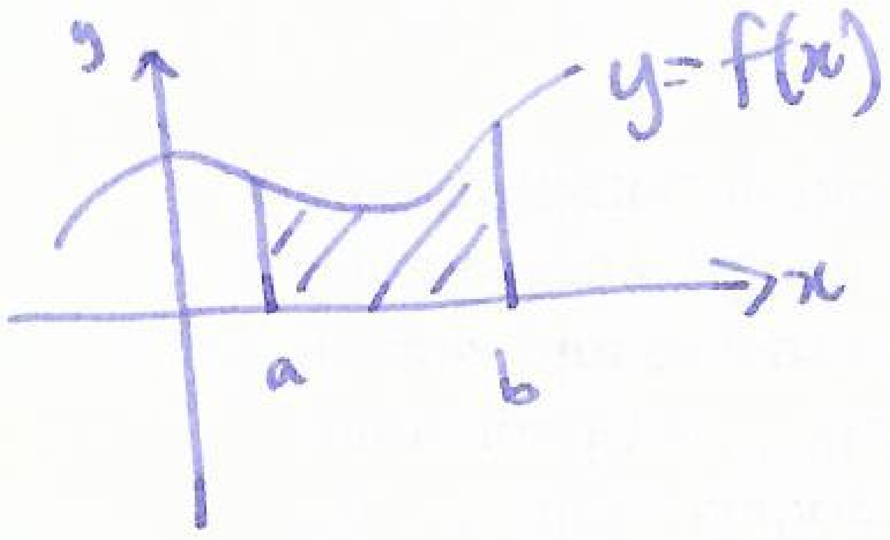
$$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{\ln(b)\cdot x}\right) = \ln(b)\,e^{\ln(b)\cdot x} = \ln(b)\,b^x.$$

so $\int b^x\,dx = \frac{1}{\ln(b)}\,b^x + c$. (or just do $\int e^{\ln(b)x}\,dx$).

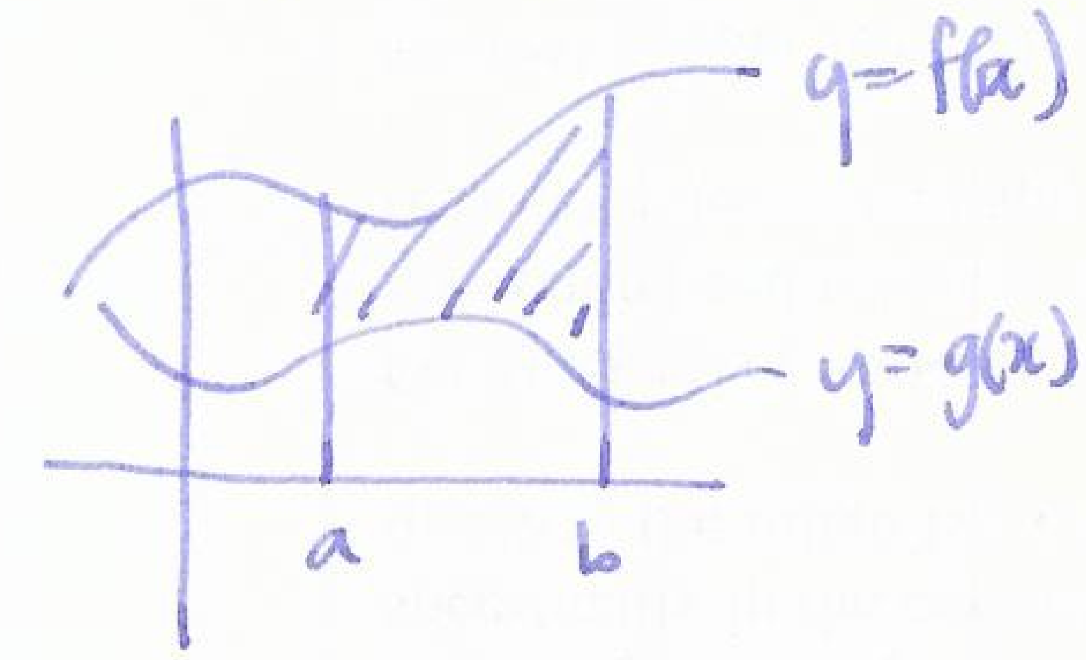
Examples $\int_0^1 10^x\,dx \qquad \int_0^4 e^{4x}\,dx \qquad \int_0^{\pi/2} (\cos\theta)\,10^{\sin\theta}\,d\theta$.

§6.1 Area between two curves

Recall

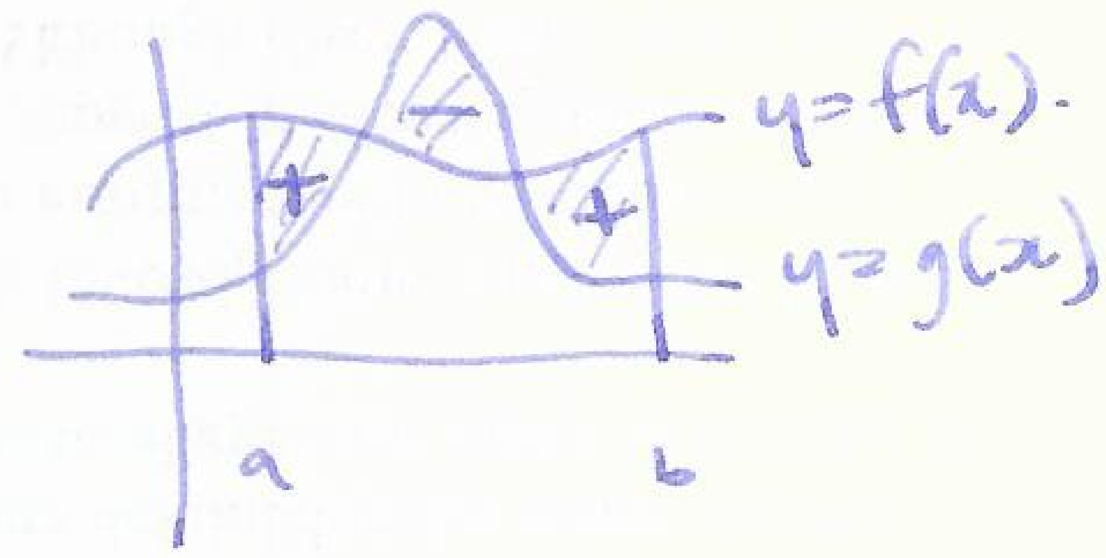


area under graph $= \int_a^b f(x)\,dx$



area between two curves $= \int_a^b f(x) - g(x)\,dx$.

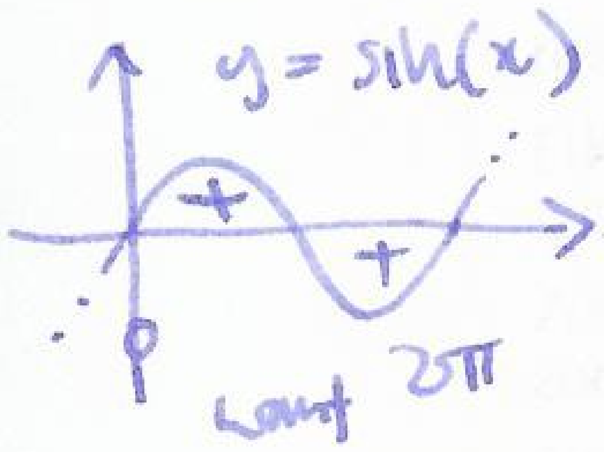
warning! signs!



$\int_a^b f(x) - g(x)\,dx$

↑ gives areas with signs!

Q: what do you do if you want "absolute value of area"?

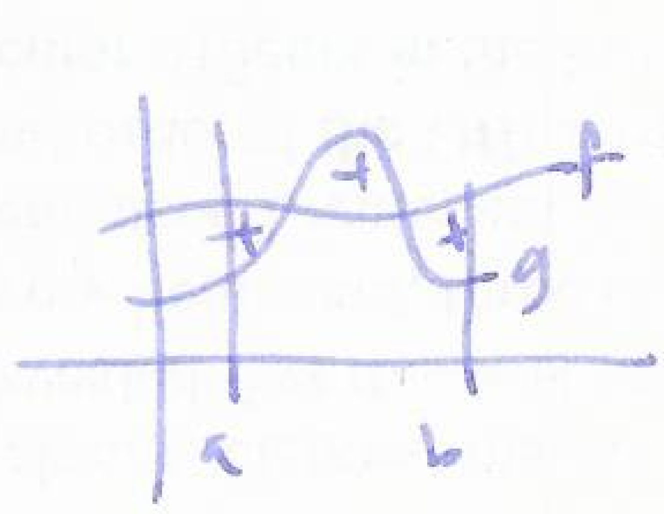


"$\int_0^{2\pi} |\sin(x)|\,dx$" ← need to know when sin(x) +/−ve.

$\int_0^{\pi} \sin(x)\,dx - \int_{\pi}^{2\pi} \sin(x)\,dx$.

i.e. need to find zeros, or sol^ns to $f(x) = 0$.

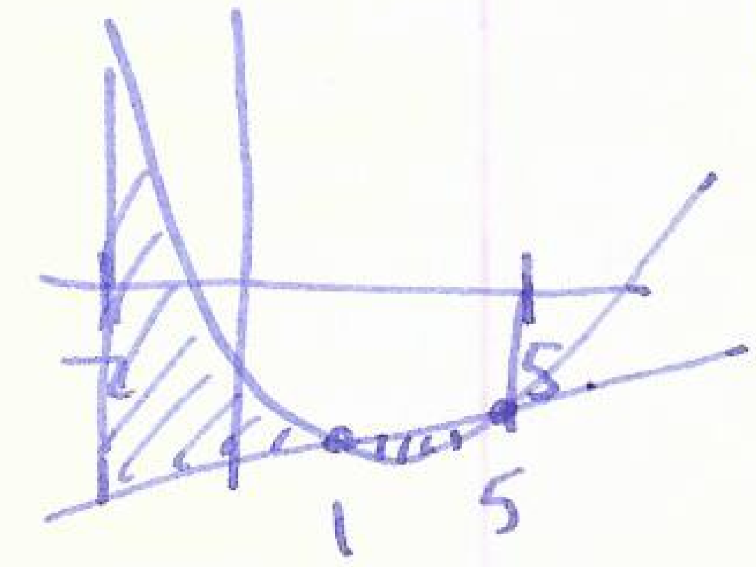
in this case:



need to find all sol^ns to $f(x) = g(x)$ in $[a, b]$.

Example - Find the area between the graphs $f(x) = x^2 - 5x - 7$ on $[-2, 5]$.
$g(x) = x - 12$

$\frac{113}{3}$



- Find the area of the region bounded by

$y = \frac{1}{x^2}$, $y = x$, $y = 8x$