Math 232 Calculus 2 Fall 12 Midterm 2b

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no phones.
- You may use a 3×5 inch index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) Find $\int_0^{\pi/4} \cos^5 x \, dx. = \int_0^{\pi/4} \cos x \left(1 - \sin^2 x\right)^2 dx$

$u = \sin x$

$\frac{du}{dx} = \cancel{\sin} \cos(x)$

$$\int_0^{\sqrt{2}/2} \cos x \; (1-u^2)^2 \frac{dx}{du} \, du = \int_0^{\sqrt{2}/2} \cos(x) \, (1-u^2)^2 \frac{1}{\cos(x)} \, du.$$

$$= \int_0^{\sqrt{2}/2} 1 - 2u^2 + u^4 \, du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \right]_0^{\sqrt{2}/2} = \frac{\sqrt{2}}{2} - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{2}}{2}\right)^5.$$

(2) Find $\int \sin(3x)\cos(5x)\;dx$.

$\sin(A+B) = \sin A\cos B + \cos B \sin A$

$\sin(A-B) = \sin A\cos B - \cos A\sin B.$

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B.$

$= \frac{1}{2}\int \sin(8x) + \sin(-2x)\;dx = -\frac{1}{8}\frac{1}{2}\cos(8x) + \frac{1}{2}\frac{1}{2}\cos(-2x) + C$

(3) Find $\int \sqrt{9 - x^2}\, dx$.

$\cos^2 u + \sin^2 u = 1$

$\cancel{\sin \cos^2 u = 1 - \cos^2 u}$.

$\cos^2 u = 1 - \sin^2 u$

$x = \cancel{3\cos u}$ $3\sin u$ $\quad \frac{dx}{du} = \cancel{-3\sin u}$. $3\cos u$.

$\cancel{\int \sqrt{9 - 9\cos^2 u}\, \frac{dx}{du}\, du = -\int 3}$

$$\int \sqrt{9 - 9\sin^2 u} \cdot \frac{dx}{du}\, du = \int 3\cos u\, 3\cos u\, du = 9\int \cos^2 u\, du$$

$\cos 2u = \cos^2 u - \sin^2 u$

$= 2\cos^2 u - 1$

$$= \frac{9}{2}\int \cos 2u + 1\, du$$

$$= \frac{9}{2}\left[\frac{1}{2}\sin 2u + u\right] = \frac{9}{4}\sin 2u + \frac{9u}{2} + c$$

(4) Find $\int \frac{3}{x^2 - 9}\, dx$.

$$\frac{3}{x^2-9} = \frac{3}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} = \frac{A(x+3) + B(x-3)}{(x-3)(x+3)}.$$

$x = 3$: $\quad 3 = 6A \quad A = 1/2$

$x = -3$: $\quad 3 = -6B \quad B = -1/2$.

$$\int \frac{1/2}{x-3} - \frac{1/2}{x+3}\, dx = \frac{1}{2} \ln|x-3| - \frac{1}{2}\ln|x+3| + c$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

(5) Find $\int_1^\infty xe^{-3x}\,dx$. $= \lim_{R\to\infty} \int_1^R \underbrace{x}_{u}\underbrace{e^{-3x}}_{v'}\,dx$

$u = x \qquad u' = 1$

$v' = e^{-3x} \qquad v = -\frac{1}{3}e^{-3x}$

$$= \lim_{R\to\infty} \left[-\frac{1}{3}xe^{-3x}\right]_1^R - \int_1^R -\frac{1}{3}e^{-3x}\,dx$$

$$= \lim_{R\to\infty} \underbrace{-\frac{1}{3}Re^{-3R}}_{\to 0} + \frac{1}{3} + \left[-\frac{1}{9}e^{-3x}\right]_1^R.$$

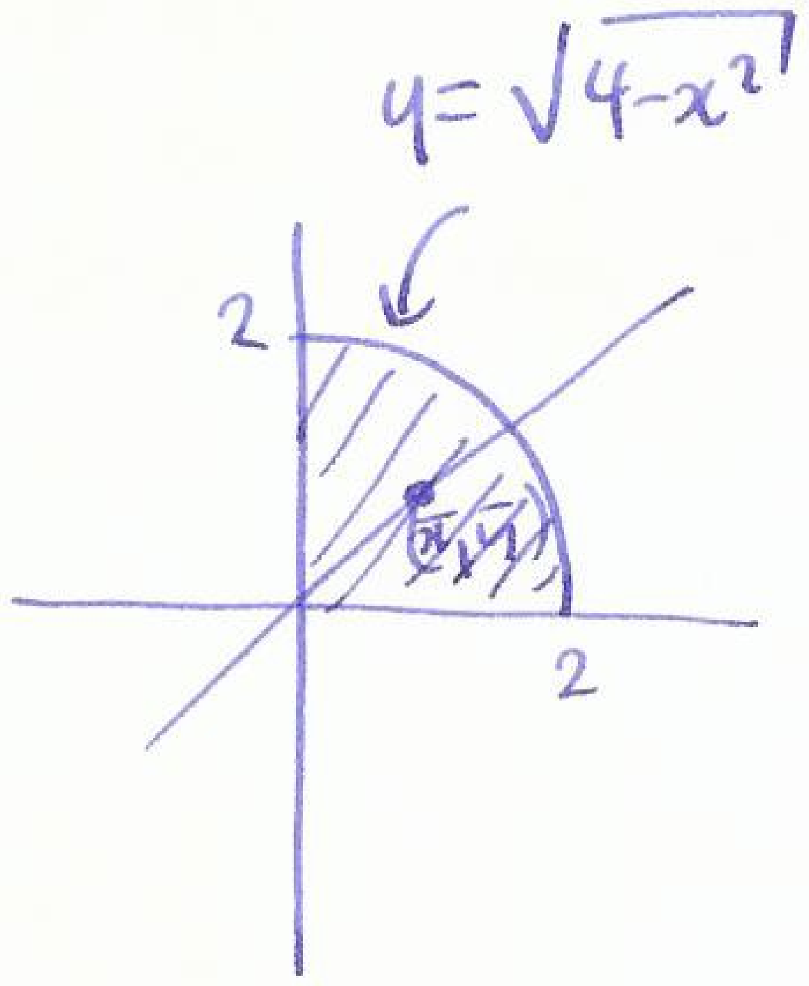
$$= \frac{1}{3}e^{-3} + \lim_{R\to\infty} \underbrace{-\frac{1}{9}e^{-3R}}_{\to 0} + \frac{1}{9}e^{-3} = e^{-3}\left(\frac{1}{3}+\frac{1}{9}\right) = \frac{4}{9}e^{-3}$$

(6) Find $\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$. $= \lim_{R\to 0} \int_R^1 x^{-1/3}\,dx$

$= \lim_{R\to 0} \left[\frac{3x^{2/3}}{2} \right]_R^1 = \lim_{R\to 0} \frac{3}{2} - \frac{3}{2} R^{2/3} = \frac{3}{2}$

(7) Find the center of mass of the region with constant density $\rho = 1$ inside the circle $x^2 + y^2 = 4$, and in the first quadrant, i.e. $x \geq 0$ and $y \geq 0$.

Hint: You may use the fact that the area of the whole disc is πr^2, and, by symmetry, you may assume that the center of mass lies on the line $y = x$.



$$M = \rho \cdot \text{area} = \underbrace{\rho}_{=1} \cdot \frac{\pi (2)^2}{4} = \pi$$

$$M_x = \int_0^2 x\sqrt{4-x^2}\, dx \qquad u = 4-x^2 \qquad \frac{du}{dx} = -2x$$

$$M_x = \int_4^0 x\, u^{1/2} \frac{dx}{du}\, du = \int_4^0 x \cdot u^{1/2} \cdot \frac{1}{-2x}\, du = -\frac{1}{2}\int_4^0 u^{1/2}\, du$$

$$= \frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_0^4 = \frac{1}{2} \cdot \frac{2}{3} \cdot 4^{3/2} = \frac{1}{3} 4^{3/2}.$$

$$\bar{x} = \frac{4^{3/2}}{3\pi} = \bar{y} \qquad \text{so center of mass } \left(\frac{4^{3/2}}{3\pi}, \frac{4^{3/2}}{3\pi}\right).$$

$$\left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$$

(8) Find the degree three Taylor polynomial for the function $f(x) = \cos(x^2)$.

$f(x) = \cos(x^2)$ $\quad f(0) = 1$

$f'(x) = -\sin(x^2).2x$ $\quad f'(0) = 0$

$f''(x) = -\sin(x^2).2 - \cos(x^2).4x^2$ $\quad f''(0) = 0$

$f'''(x) = -2\cos(x^2).2x + \sin(x^2).8x^3 - \cos(x^2).8x.$ $\quad f^{(3)}(0) = 0$

$T_3(x) = 1$

(9) Does the series $\sum_{n=2}^{\infty} \frac{1}{n^2+n}$ converge or diverge? Justify your answer.

$n^2+n > n^2 \Rightarrow \frac{1}{n^2+n} < \frac{1}{n^2}$

comparison test: $0 < \frac{1}{n^2+n} < \frac{1}{n^2}$, $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (p-series or integral test)

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2+n}$ converges

(10) Does the series $\sum_{n=2}^{\infty} \frac{n}{9^n}$ converge or diverge? Justify your answer.

$n \leq 3^n$ for $n \geq 1$ so $0 \leq \frac{n}{9^n} \leq \frac{3^n}{9^n} \leq \frac{1}{3^n}$

comparison test : $\sum_{n=2}^{\infty} \frac{1}{3^n}$ converges (geometric series or integral test)

$\Rightarrow \sum_{n=2}^{\infty} \frac{n}{9^n}$ converges.