Sample Midterm 2  Solutions



**Q1** $\displaystyle\int_0^{\pi/2} \sin^3(x)\cos^2(x)\,dx$   try: $u = \cos(x)$, $\displaystyle\frac{du}{dx} = -\sin(x)$.

$$= \int_1^0 \sin(x)\left(1-u^2\right)u^2 \underbrace{\frac{dx}{du}}_{= -\frac{1}{\sin(x)}}\,du = \int_1^0 u^4 - u^2\,du = \left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^0$$

$$= -\frac{1}{5} + \frac{1}{3} = \frac{2}{15}.$$

**Q2**
$$\left.\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ \cos(A-B) &= \cos A\cos B + \sin A\sin B \end{aligned}\right\} \cos(A+B) + \cos(A-B) = 2\cos A\cos B$$

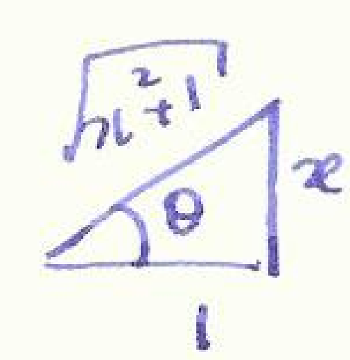
$$\int \cos(3x)\cos(5x)\,dx = \frac{1}{2}\int \cos(8x) + \cos(2x)\,dx = \frac{1}{16}\sin(8x) + \frac{1}{4}\sin(2x) + C.$$

**Q3** $\displaystyle\int \frac{x^2}{\sqrt{x^2+1}}\,dx$   try: $x = \tan\theta$, $\displaystyle\frac{dx}{d\theta} = \sec^2\theta$   $\displaystyle\int \frac{\tan^2\theta}{\sqrt{\tan^2\theta+1}} \cdot \frac{dx}{d\theta}\,d\theta$

$$= \int \frac{\tan^2\theta}{\sec\theta} \cdot \sec^2\theta\,d\theta = \int \tan^2\theta\sec\theta\,d\theta$$

$u = \tan\theta$   $u' = \sec^2\theta$
$v' = \tan\theta\sec\theta$   $v = \sec\theta$

$$\int \underbrace{\tan^2\theta\sec\theta}_{uv'}\,d\theta = \underbrace{\tan\theta\sec\theta}_{uv} - \int \underbrace{\sec^2\theta \cdot \sec\theta}_{u'v}\,d\theta$$
$$= \tan\theta\sec\theta - \int \left(1+\tan^2\theta\right)\sec\theta\,d\theta$$
$$= \tan\theta\sec\theta - \int \sec\theta\,d\theta - \int \tan^2\theta\sec\theta\,d\theta.$$




$$2\int \tan^2\theta\sec\theta\,d\theta = \tan\theta\sec\theta - \ln\left|\sec\theta + \tan\theta\right| + c'$$

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\,x \cdot \frac{1}{\sqrt{x^2+1}} - \frac{1}{2}\ln\left|\frac{1}{\sqrt{x^2+1}} + x\right| + c$$

**Q4**
$$\frac{x^2+11x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}.$$

$x=1$: $12 = 4A \Rightarrow A = 3.$   $x=0$: $0 = A - B - C = 3 - B - 5 \Rightarrow B = -2$

$x=-1$: $-10 = -2C \Rightarrow C = 5$

$$\int \frac{x^2+11x}{(x-1)(x+1)^2}\,dx = \int \frac{3}{x-1} + \frac{-2}{x+1} + \frac{5}{(x+1)^2}\,dx = 3\ln|x-1| - 2\ln|x+1| - \frac{5}{2}(x+1)^{-1} + c$$

**Q5** $\int_0^1 \underbrace{x}_{v'}\, \underbrace{\ln(x)}_{u}\, dx$ $\qquad \int uv'\,dx = uv - \int u'v\,dx$ $\qquad u = \ln(x) \quad u' = \frac{1}{x}$ $\qquad v' = x \quad v = \frac{1}{2}x^2$

$$\lim_{R\to 0}\int_R^1 x\ln(x)\,dx = \left[\frac{1}{2}x^2\ln(x)\right]_R^1 - \int_R^1 \frac{1}{x}\cdot\frac{1}{2}x^2\,dx = -\frac{1}{2}R^2\ln(R) - \int_R^1 \frac{1}{2}x\,dx$$

$$\lim_{R\to 0} -\frac{1}{2}R^2\ln(R) - \left[\frac{1}{4}x^2\right]_R^1 = \lim_{R\to 0} -\frac{1}{2}R^2\ln(R) - \frac{1}{4} + \underbrace{\frac{1}{4}R^2}_{\to 0.}$$

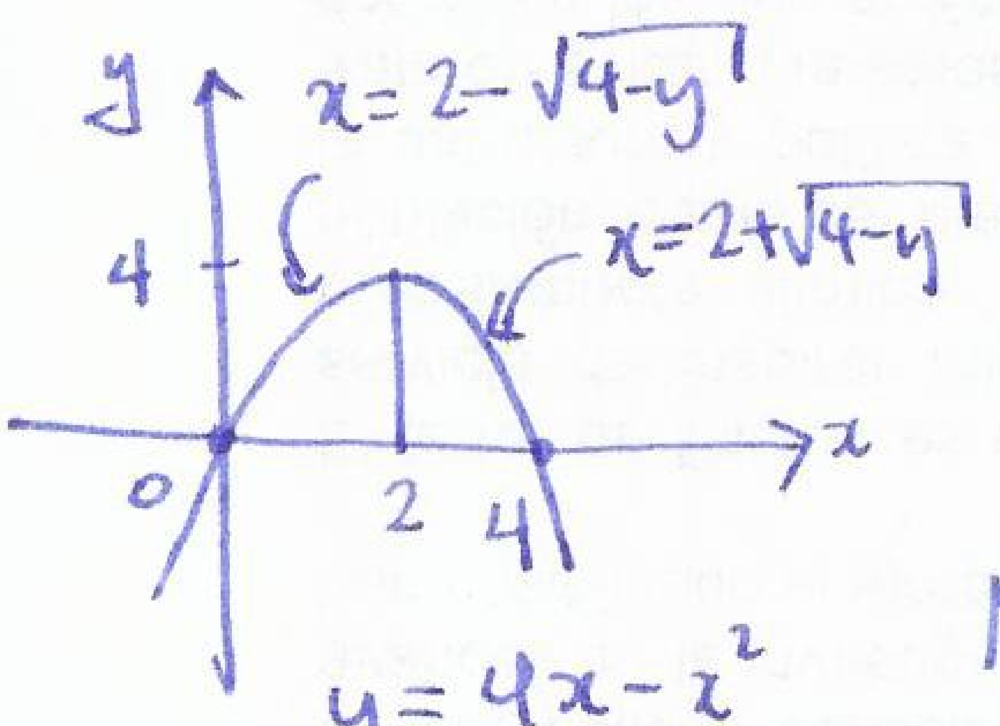
$$= -\frac{1}{4} - \frac{1}{2}\lim_{R\to 0}\frac{\ln(R)}{1/R^2} \overset{\text{l'Hôpital}}{=} -\frac{1}{4} - \frac{1}{2}\lim_{R\to 0}\frac{1/R}{-2R^{-3}} = -\frac{1}{4} - \frac{1}{2}\underbrace{\lim_{R\to 0} -\frac{1}{2}R^2}_{=0} = -\frac{1}{4}.$$

**Q6** $$\int_0^\infty \frac{1}{4+x^2}\,dx = \lim_{R\to\infty}\int_0^R \frac{1}{4+x^2}\,dx = \cancel{\lim_{R\to\infty}} \qquad u = 2x \quad \frac{du}{dx} = 2$$

$$= \lim_{R\to\infty}\int_0^{2R} \frac{1}{4+4u^2}\cdot\frac{dx}{du}\,du = \lim_{R\to\infty}\frac{1}{8}\int_0^{2R}\frac{1}{1+u^2}\,du = \frac{1}{8}\lim_{R\to\infty}\left[\tan^{-1}(u)\right]_0^{2R}$$

$$= \frac{1}{8}\lim_{R\to\infty}\tan^{-1}(2R) - 0 = \frac{\pi}{16}.$$

**Q7**




$$M = \rho\,\text{area} = \rho\int_0^4 4x - x^2\,dx = \rho\left[2x^2 - \frac{x^3}{3}\right]_0^4$$

$$= \rho\left(32 - \frac{64}{3}\right) = \rho\frac{32}{3}$$

$$M_x = \rho\int_0^4 x(4x-x^2)\,dx = \rho\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^4$$

$x^2 - 4x + y = 0$

$x = \frac{4 \pm \sqrt{16-4y}}{2} = 2 \pm \sqrt{4-y}$

$$= \rho\left(\frac{256}{3} - \frac{256}{4}\right) = \rho\frac{256}{12} = \frac{64}{3}\rho. \qquad \bar{x} = \frac{M_x}{M} = \frac{\frac{64}{3}\rho}{32/3\rho} = 2$$

$$M_y = \rho\int_0^4 y\left((2+\sqrt{4-y}) - (2-\sqrt{4-y})\right)dy = \rho\int_0^4 \underbrace{2y}_{u}\underbrace{\sqrt{4-y}}_{v'}\,dy \qquad u = 2y \quad u' = 2 \qquad v' = \sqrt{4-y} \quad v = \frac{-2}{3}(4-y)^{3/2}$$

$$= \rho\underbrace{\left[2y\cdot\frac{-2}{3}(4-y)^{3/2}\right]_0^4}_{=0} + \rho\frac{4}{3}\int_0^4 (4-y)^{3/2}\,dy = \left[-\rho\frac{4}{3}\frac{2}{5}(4-y)^{5/2}\right]_0^4 = \frac{8}{15}\rho\,4^{5/2}$$

$$\bar{y} = \frac{M_y}{M} = \frac{\frac{8}{15}\rho\,4^{5/2}}{\rho\,32/3} = \frac{1}{20}4^{5/2} = \frac{1}{5}4^{3/2}.$$

$T_3(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$

$a_n = \dfrac{f^{(n)}(1)}{n!}$

$f(x) = \sqrt{2x+1}$ $\qquad f(1) = \sqrt{3}$

$f'(x) = \frac{1}{2}(2x+1)^{-1/2} \cdot 2$ $\qquad f'(1) = \frac{1}{\sqrt{3}}$

$f''(x) = -\frac{1}{2}(2x+1)^{-3/2} \cdot 2$ $\qquad f''(1) = -\frac{1}{3^{3/2}}$

$f^{(3)}(x) = \frac{3}{2}(2x+1)^{-5/2}$ $\qquad f^{(3)}(1) = \frac{3}{2} \cdot 3^{-5/2}$

$$T_3(x) = \sqrt{3} + \frac{1}{\sqrt{3}}(x-1) + \frac{-1}{2 \cdot 3^{3/2}}(x-1)^2 + \frac{3^{-3/2}}{2 \cdot 6}(x-1)^3.$$

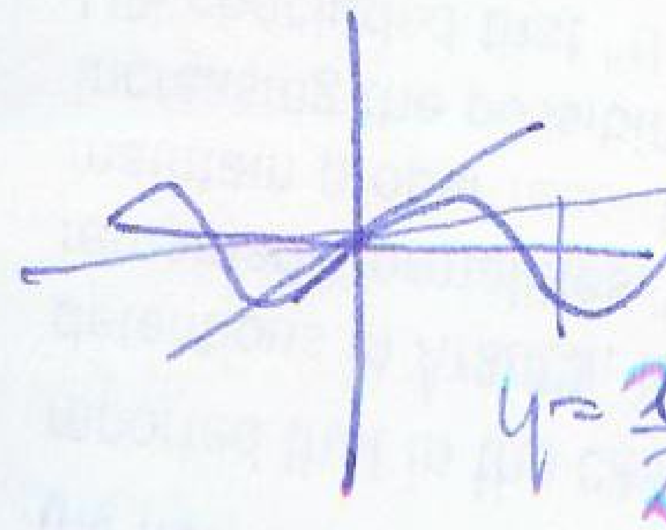
Q9 comparison test. $\frac{\ln(n)}{n^3}$ positive sequence

$\ln(n) \leq n$.

$$0 \leq \frac{\ln(n)}{n^3} \leq \frac{1}{n^2}$$

as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$ converges.

Q10



$y = \frac{x}{2}$.

$\sin(x) \sim x$ for $x$ close to 0

comparison test $\qquad 0 \leq \frac{1}{2n} \leq \sin\left(\frac{1}{n}\right)$

$\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges.