Calc 2 Sample midterm 1 Solutions

Q1 $\int x \sin(-2x^2)\,dx$

$u = -2x^2$

$\frac{du}{dx} = -4x$

$$\int x \sin(u) \frac{dx}{du}\,du = \int x \sin(u) \frac{1}{-4x}\,du = -\frac{1}{4}\int \sin(u)\,du$$

$$= +\frac{1}{4}\cos(u) + c = \frac{1}{4}\cos(-2x^2) + c$$

Q2 $\int \frac{1}{2}x^2(1-x^3)^{1/3}\,dx$

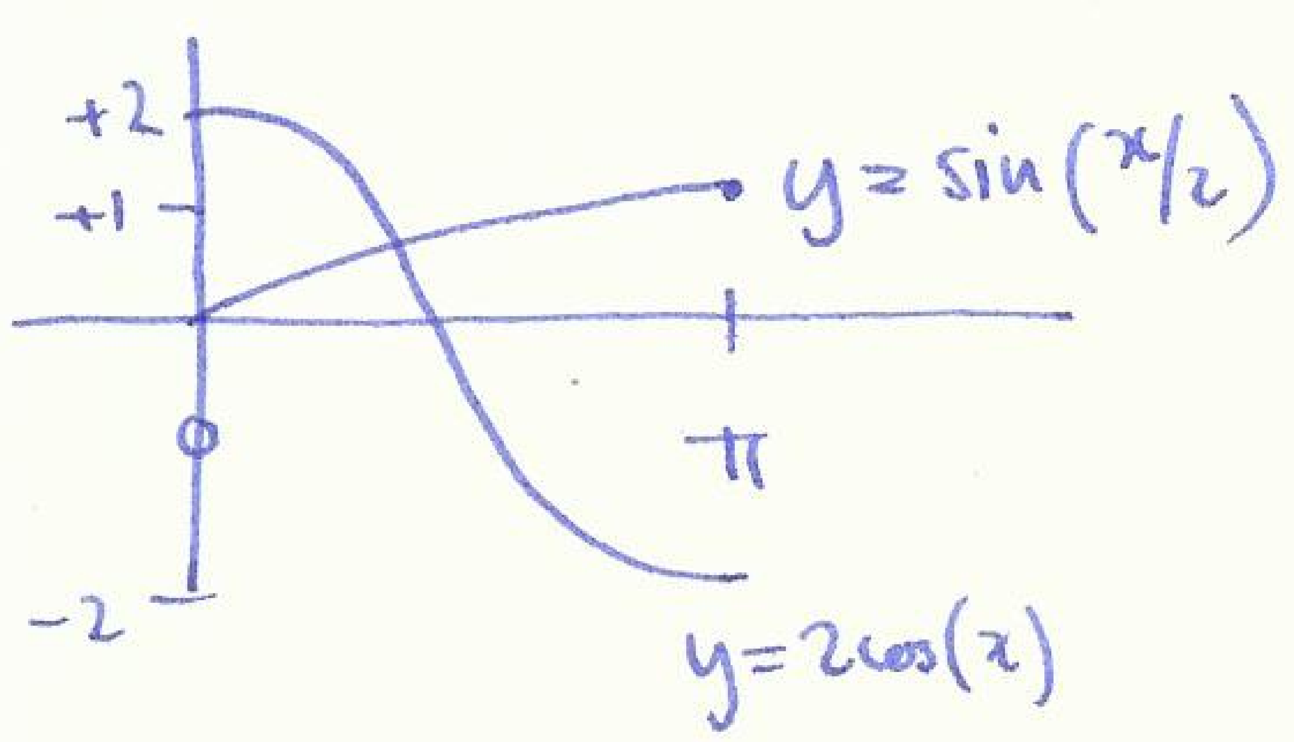
$u = 1-x^3$

$\frac{du}{dx} = -3x^2$

$$\int \frac{1}{2}x^2 (u)^{1/3} \frac{dx}{du}\,du = \int \frac{1}{2}x^2 (u)^{1/3} \frac{1}{-3x^2}\,du = -\frac{1}{6}\int u^{1/3}\,du$$

$$= -\frac{3}{24}u^{4/3} + c = -\frac{1}{8}u^{4/3} + c = -\frac{1}{8}(1-x^3)^{4/3} + c$$

Q3



find intersection point: $\sin(\frac{x}{2}) = 2\cos(x)$

double angle formula: $\cos(x) = \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})$

$= 1 - 2\sin^2(\frac{x}{2})$.

so $\sin(\frac{x}{2}) = 2 - 4\sin^2(\frac{x}{2})$

quadratic in $\sin(\frac{x}{2})$ $\rightarrow$ $4\sin^2(\frac{x}{2}) + \sin(\frac{x}{2}) - 2 = 0$

$$\sin(\tfrac{x}{2}) = \frac{-1 \pm \sqrt{1 - 4\cdot 4\cdot(-2)}}{2\cdot 4}$$

$$\sin(\tfrac{x}{2}) = \frac{-1 \pm \sqrt{33}}{8}$$

$$x = 2\sin^{-1}\left(\frac{-1 + \sqrt{33}}{8}\right) \approx 1.270$$

area = $\int_0^a 2\cos(x) - \sin(\frac{x}{2})\,dx + \int_a^\pi \sin(\frac{x}{2}) - 2\cos(x)\,dx$ $\quad a = 1.270$

$= \left[2\sin(x) + 2\cos(\frac{x}{2})\right]_0^a + \left[-2\cos(\frac{x}{2}) - 2\sin(x)\right]_a^\pi$

$= 2\sin(a) + 2\cos(\frac{a}{2}) - 2 + 2\cos(\frac{a}{2}) + 2\sin(a)$

$= 4\sin(a) + 4\cos(\frac{a}{2}) - 2 \approx 5.041$

Q4 a) cross section at height z: $x^2+y^2 = \frac{1-z^2}{4}$ circle of radius $r^2 = \frac{1-z^2}{4}$.

so area $A(z) = \pi\left(\frac{1-z^2}{4}\right)$.

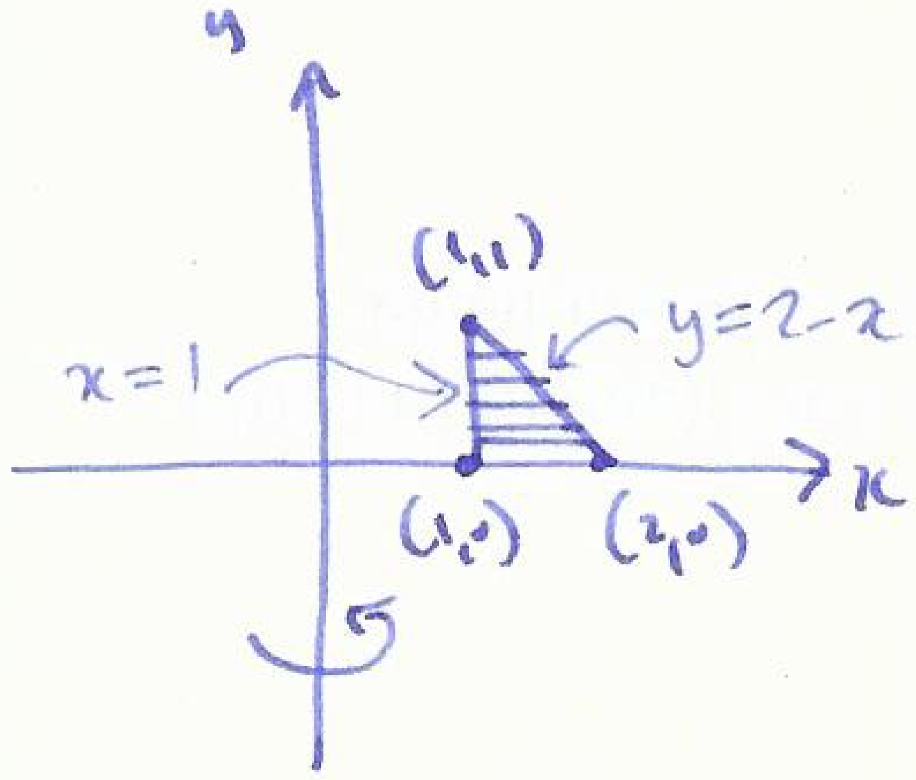
b) $V = \int_{-1}^{+1} A(z)\,dz = \int_{-1}^{1} \pi\left(\frac{1-z^2}{4}\right)dz = \frac{\pi}{4}\int_{-1}^{1} 1-z^2\,dz = \frac{\pi}{4}\left[z - \frac{1}{3}z^3\right]_{-1}^{1}$

$= \frac{\pi}{4}\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right] = \frac{\pi}{2}\left(1-\frac{1}{3}\right) = \frac{\pi}{2}\frac{2}{3} = \frac{\pi}{3}$.

Q5 $\frac{1}{4}\int_{-1}^{3} e^{-x/2}\,dx$ $\quad u = -x/2$ $\quad \frac{du}{dx} = -\frac{1}{2}$ $\quad \frac{1}{4}\int_{1/2}^{-3/2} e^u(-2)\,du$

$= -\frac{1}{2}\left[e^u\right]_{1/2}^{-3/2} = -\frac{1}{2}\left(e^{-3/2} - e^{1/2}\right) = \frac{1}{2}\left(e^{1/2} - e^{-3/2}\right)$

Q6



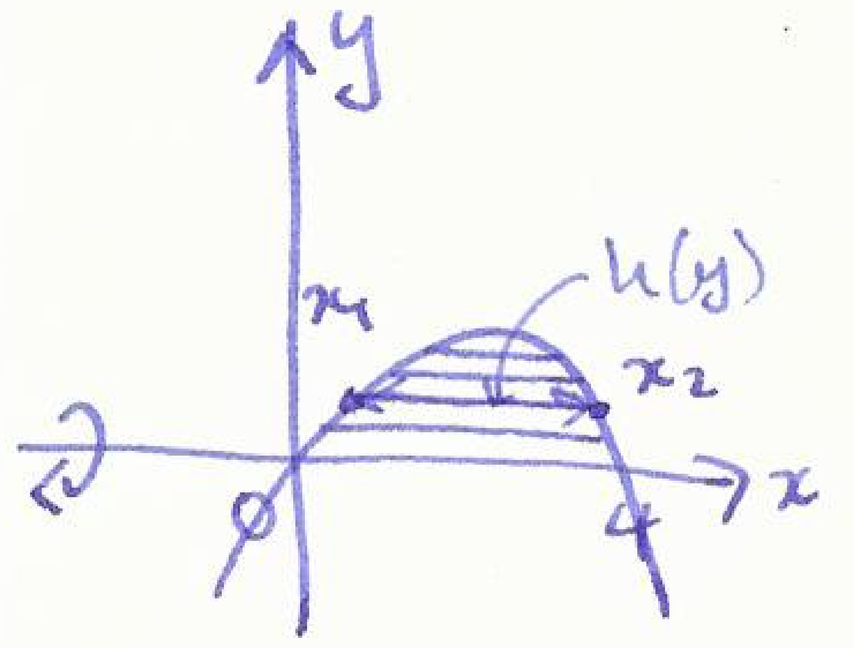
$V = \int_0^1 \pi(r_2^2 - r_1^2)\,dy$ $\quad r_1 = 1$ $\quad r_2 = 2-y$

$V = \int_0^1 \pi\left(-1 + (2-y)^2\right)dy$

$$V = \pi \int_0^1 4 - 4y + y^2 - 1 \, dy = \pi \int_0^1 3 - 4y + y^2 \, dy = \pi \left[3y - 2y^2 + \tfrac{1}{3} y^3 \right]_0^1$$

$$= \pi \left(3 - 2 + \tfrac{1}{3} \right) = \tfrac{4}{3} \pi .$$

Q7



$y = -x^2 + 4x = -(x-2)^2 + 4$ max value $y = 4$.

$$V = \int_0^4 2\pi y h(y) \, dy$$

find $h(y)$: $y = -x^2 + 4x$

$$x^2 - 4x + y = 0 \qquad x = \frac{4 \pm \sqrt{16 - 4y}}{2} = 2 \pm \sqrt{4 - y} \qquad x_1 = 2 - \sqrt{4-y}, \quad x_2 = 2 + \sqrt{4-y}.$$

$$h(y) = x_2 - x_1 = 2 + \sqrt{4-y} - (2 - \sqrt{4-y}) = 2\sqrt{4-y}.$$

$$V = \int_0^4 2\pi y \cdot 2\sqrt{4-y} \, dy = 4\pi \int_0^4 y\sqrt{4-y} \, dy \qquad u = 4 - y, \quad \frac{du}{dy} = -1.$$

$$= 4\pi \int_4^0 (4-u) u^{1/2} \frac{dy}{du} \, du = -4\pi \int_4^0 4u^{1/2} - u^{3/2} \, du = -4\pi \left[\frac{8}{3} u^{3/2} - \frac{2u^{5/2}}{5} \right]_4^0$$

$$= 4\pi \left(\frac{8 \cdot 4^{3/2}}{3} - \frac{2 \cdot 4^{5/2}}{5} \right).$$

Q8 error: $\dfrac{K_2 (b-a)^3}{12 N^2}$ $\quad [a,b] = [0,2]$

$$f(x) = x e^{-2x}$$

$$f'(x) = e^{-2x} + x \cdot (-2e^{-2x})$$

$$f''(x) = -2e^{-2x} + (-2e^{-2x}) + x(4e^{-2x}) \quad \cancel{\text{st } |f''(x)| \leq}$$

so $|f''(x)| \leq 2 \cdot e^4 + 2e^4 + 2 \cdot 4e^4 = 12e^4$

$$\text{error}(T_{10}) \leqslant \frac{12e^4 \cdot 2^3}{12 \cdot 10^2}$$

4 decimal places: $$\frac{12e^4 \cdot 2^3}{12 \cdot N^2} < \frac{1}{2} 10^{-4}$$

$$N^2 > \cancel{4}e^4 \cdot 2^3 \cdot 2 \, 10^4$$

Q9 $$\int \underbrace{x^2}_{v'} \underbrace{\ln(x)}_{u} \, dx \qquad \int u v' \, dx = uv - \int u' v \, dx$$

$$v' = x^2 \qquad v = \frac{1}{3}x^3$$

$$u = \ln(x) \qquad u' = \frac{1}{x}$$

$$\int x^2 \ln(x) \, dx = \frac{1}{3}x^3 \ln(x) - \int \frac{1}{3}x^2 \, dx = \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C.$$

Q10 $$\int \underbrace{e^{-2x}}_{u} \underbrace{\cos(3x)}_{v'} \, dx = e^{-2x} \cdot \frac{1}{3}\sin(3x) - \int (-2e^{-2x}) \cdot \frac{1}{3}\sin(3x) \, dx$$

$$u = e^{-2x} \qquad u' = -2e^{-2x}$$

$$v' = \cos(3x) \qquad v = \frac{1}{3}\sin(3x)$$

$$\int e^{-2x} \cos(3x) \, dx = \frac{1}{3}e^{-2x}\sin(3x) + \frac{2}{3}\int \underbrace{e^{-2x}}_{u} \underbrace{\sin(3x)}_{v'} \, dx$$

$$u = e^{-2x} \qquad u' = -2e^{-2x}$$

$$v' = \sin(3x) \qquad v = -\frac{1}{3}\cos(3x)$$

$$\int e^{-2x} \cos(3x) \, dx = \frac{1}{3}e^{-2x}\sin(3x) + \frac{2}{3}e^{-2x} \cdot \left(-\frac{1}{3}\cos(3x)\right) - \frac{2}{3}\int (-2e^{-2x}) \cdot \left(-\frac{1}{3}\cos(3x)\right) dx$$

$$\left(1 + \frac{4}{9}\right) \int e^{-2x} \cos(3x) \, dx = \frac{1}{3}e^{-2x}\sin(3x) - \frac{2}{9}e^{-2x}\cos(3x) + c.$$

$$\int e^{-2x} \cos(3x) \, dx = \frac{9}{13}\left(\frac{1}{3}e^{-2x}\sin(3x) - \frac{2}{9}e^{-2x}\cos(3x)\right) + c.$$