Math 232 Calculus 2 Fall 12 Final a

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no phones.
- You may use a 3×5 inch index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Final	
Overall	

(1) (a) Find $\displaystyle\int \frac{x}{x-1}\,dx$.

$$\begin{array}{r} 1 \\ x-1\,\overline{)\,x} \\ \underline{x-1} \\ +1 \end{array} \qquad \frac{x}{x-1} = 1 + \frac{1}{x-1}$$

$$\int 1 + \frac{1}{x-1}\,dx = x + \ln|x-1| + c$$

(b) Find $\displaystyle\int \frac{1}{4-x^2}\,dx$.

$$\frac{1}{4-x^2} = \frac{A}{2-x} + \frac{B}{2+x}$$

$$= \frac{A(2+x)+B(2-x)}{(2-x)(2+x)}$$

$x=2$: $1 = 4A$

$x=-2$: $1 = 4B$

$$\int \frac{1/4}{2-x} + \frac{1/4}{2+x}\,dx = -\frac{1}{4}\ln|2-x| + \frac{1}{4}\ln|2+x| + c$$

(2) Find $\int \frac{1}{9+x^2}\,dx.$ $= \frac{1}{9}\int \frac{1}{1+\left(\frac{x}{3}\right)^2}\,dx$ $\quad u=\frac{x}{3} \quad \frac{du}{dx}=\frac{1}{3}.$

$= \frac{1}{9}\int \frac{1}{1+u^2}\cdot 3\,du = \frac{1}{3}\tan^{-1}(u)+C = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$

(3) Find $\int \sin^3(x)\ dx$. $= \int (1-\underset{\cos}{\cancel{\sin}}{}^2 x)\overset{\sin}{\cancel{\cos}}\, x\ dx$

$u = \overset{\cos}{\cancel{\sin x}}$

$\frac{du}{dx} = \underset{-\sin}{\cancel{\cos x}}$

$= \int (1-u^2) \sin x \cdot \frac{1}{-\sin x}\ du = -\int 1-u^2\ du = -u + \frac{1}{3}u^3 + C$

$= -\cos(x) + \frac{1}{3}\cos^3(x) + C$

(4) Find $\int_0^\infty xe^{-3x}\, dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

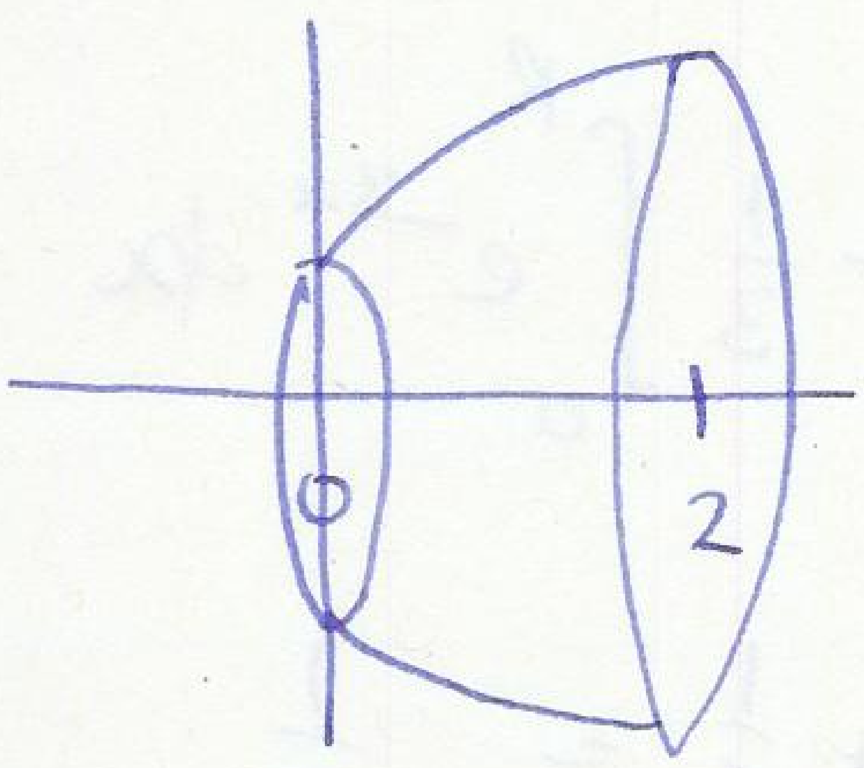
$u = x \quad u' = 1$

$v' = e^{-3x} \quad v = -\frac{1}{3}e^{-3x}$

$$= \lim_{R\to\infty} \left[x \cdot -\frac{1}{3}e^{-3x}\right]_0^R + \frac{1}{3}\int_0^R e^{-3x}\,dx$$

$$= 0 + \lim_{R\to\infty} \frac{1}{3}\left[-\frac{1}{3}e^{-3x}\right]_0^R = \lim_{R\to\infty} -\frac{1}{9}e^{-3R} + \frac{1}{9} = \frac{1}{9}.$$

(5) Find the volume of revolution of the curve $y = 1 + \sqrt{x}$ above the interval $[0, 2]$, rotated about the x-axis.



$$\int_0^2 \pi \left(1+\sqrt{x}\right)^2 dx = \pi \int_0^2 1 + 2\sqrt{x} + x \, dx$$

$$= \pi \left[x + \frac{4x^{3/2}}{3} + \frac{1}{2}x^2 \right]_0^2$$

$$= \pi \left(2 + \frac{8\sqrt{2}}{3} + 2 \right) = \pi \left(4 + \frac{8\sqrt{2}}{3} \right).$$

(6) Does the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n}$ converge? Justify your answer.

Yes geometric series converges to $\frac{1}{1+1/3} = \frac{3}{4}$

(7) Does the series $\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^3+2}$ converge? Justify your answer.

Yes. Comparison test : $\frac{\sqrt{n}}{n^3+2} \leq \frac{n}{n^3} = \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series) $\Rightarrow \sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^3+2}$ converges.

(8) Find a power series for the function $f(x) = e^x + e^{-x}$ centered at $c = 0$. What is the radius of convergence of the power series?

$f(x) = e^x + e^{-x}$ $\qquad f(0) = 2$

$f'(x) = e^x - e^{-x}$ $\qquad f'(0) = 0$

$f''(x) = e^x + e^{-x}$ $\qquad f''(0) = 2$

$f^{(n)}(x) = e^x + (-1)^n e^{-x}$ $\qquad f^{(n)}(0) = 1 + (-1)^n$

power series is $\displaystyle\sum_{n=0}^{\infty} \frac{(1+(-1)^n)x^n}{n!} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots$

ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{2x^{2n}}\right| = \lim_{n\to\infty} \left|\frac{x^2}{(n+1)(n+2)}\right| = 0$$

so radius of convergence is $R = \infty$.

(9) Find the angle between the vectors $\mathbf{v} = \langle 1, -2, 3\rangle$ and $\mathbf{w} = \langle -2, 3, 1\rangle$. Find the equation of the plane through the origin containing the two vectors.

$\underline{v} \cdot \underline{w} = \|\underline{v}\| \, \|\underline{w}\| \cos\theta$

$-2 - 6 + 3 = \sqrt{(1+4+9)(4+9+1)} \cos\theta$

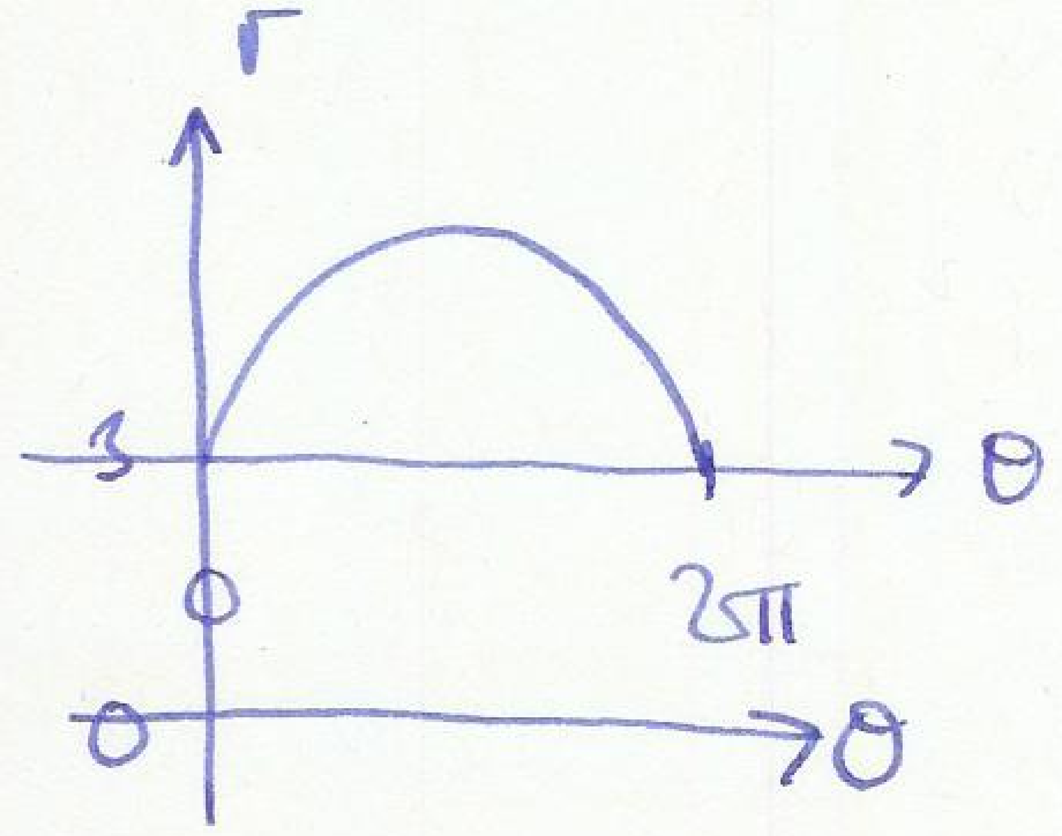
$\cos\theta = -\frac{5}{14}$

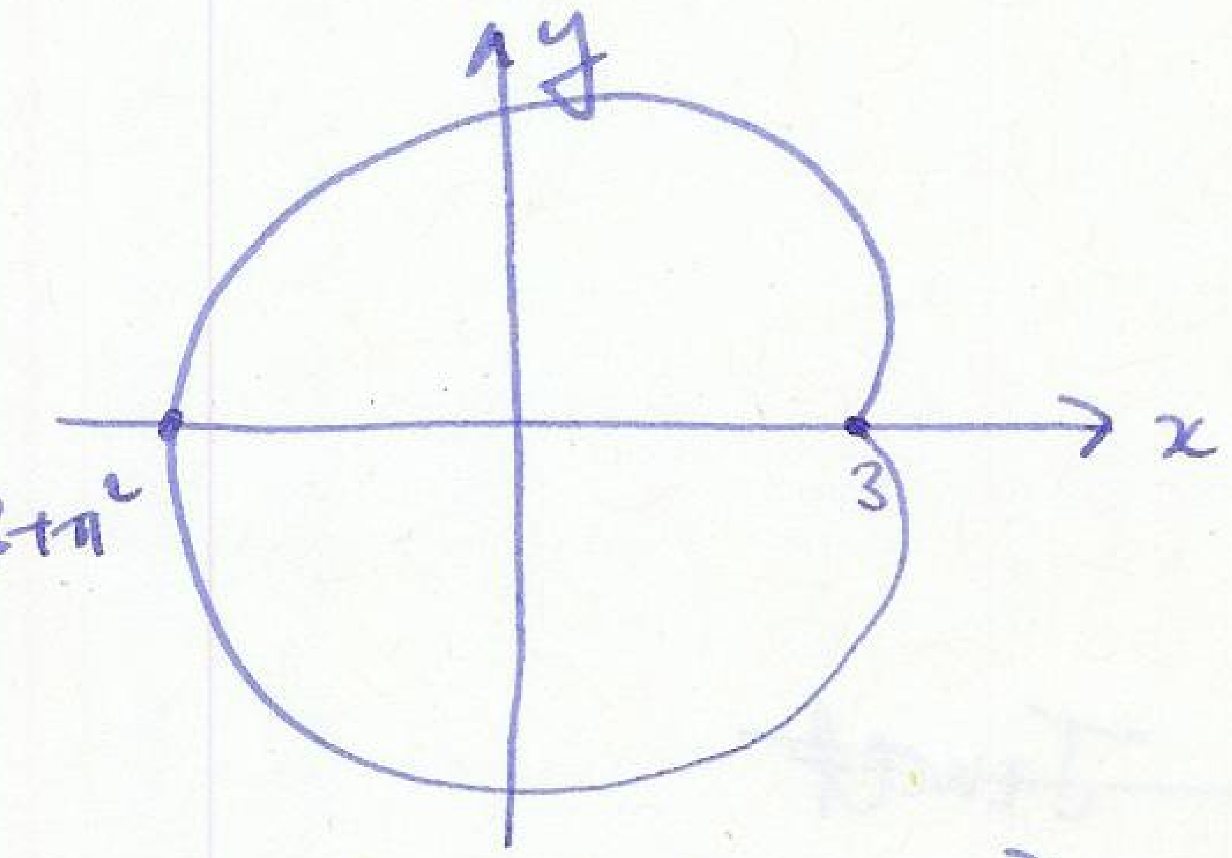
$$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 3 \\ -2 & 3 & 1 \end{vmatrix} = \underline{i} \begin{vmatrix} -2 & 3 \\ 3 & 1 \end{vmatrix} - \underline{j} \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= \langle -11, -7, -1 \rangle$$

$11x + 7y + z = 0$

(10) Sketch the polar curve $r = 3 + \theta(2\pi - \theta)$, for $0 \le \theta \le 2\pi$. Find the slope $\frac{dy}{dx}$ at $\theta = \pi/2$.





$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \qquad y = r\sin\theta = \left(3 + \theta(2\pi - \theta)\right)\sin\theta$$

$$x = r\cos\theta = \left(3 + \theta(2\pi - \theta)\right)\cos\theta.$$

$$\frac{dy}{d\theta} = 3\cos\theta + (2\pi - 2\theta)\sin\theta + \theta(2\pi - \theta)\cos\theta$$

$$\frac{dx}{d\theta} = -3\sin\theta + (2\pi - 2\theta)\cos\theta + -\theta(2\pi - \theta)\sin\theta.$$

$$\left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{2}} = \frac{0 + \pi + 0}{-3 + 0 - \frac{3\pi^2}{4}} = -\frac{\pi}{3 + \frac{3\pi^2}{4}}$$