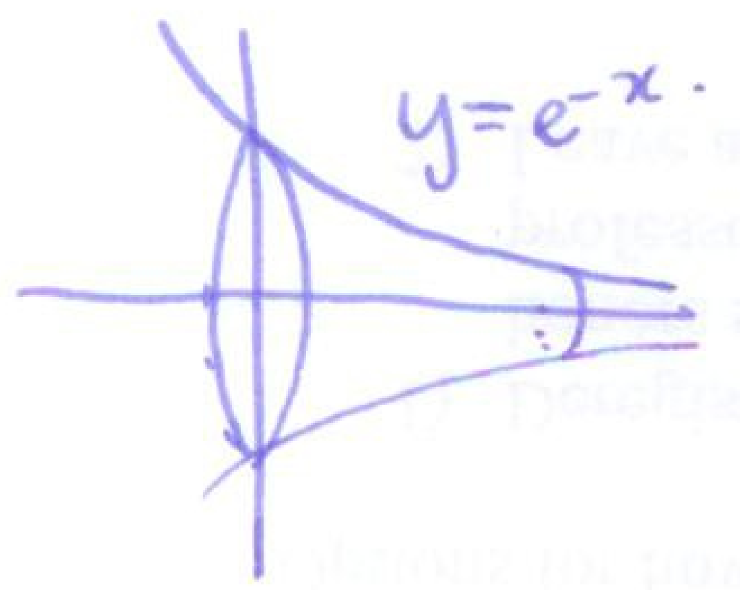
Sample final solutions

Q1



$y=e^{-x}$

$$\text{vol} = \pi\int_0^\infty \left(e^{-x}\right)^2 dx = \pi\int_0^\infty e^{-2x}\,dx$$

$$= \lim_{R\to\infty} \pi\int_0^R e^{-2x}\,dx = \lim_{R\to\infty}\left[-\tfrac{1}{2}\pi e^{-2x}\right]_0^R$$

$$= \lim_{R\to\infty} -\tfrac{1}{2}\pi e^{-2R} + \tfrac{1}{2}\pi = \tfrac{1}{2}\pi$$

Q2 a)

$f(x) = e^{-x}$ $\quad f(0) = 1 \quad a_0 = 1$

$f'(x) = -e^{-x}$ $\quad f'(0) = 1 \quad a_1 = 1$

$f''(x) = +e^{-x}$ $\quad f''(0) = 1 \quad a_2 = 1/2!$

$f^{(3)}(x) = -e^{-x}$ $\quad f^{(3)}(0) = -1 \quad a_3 = -1/3!$

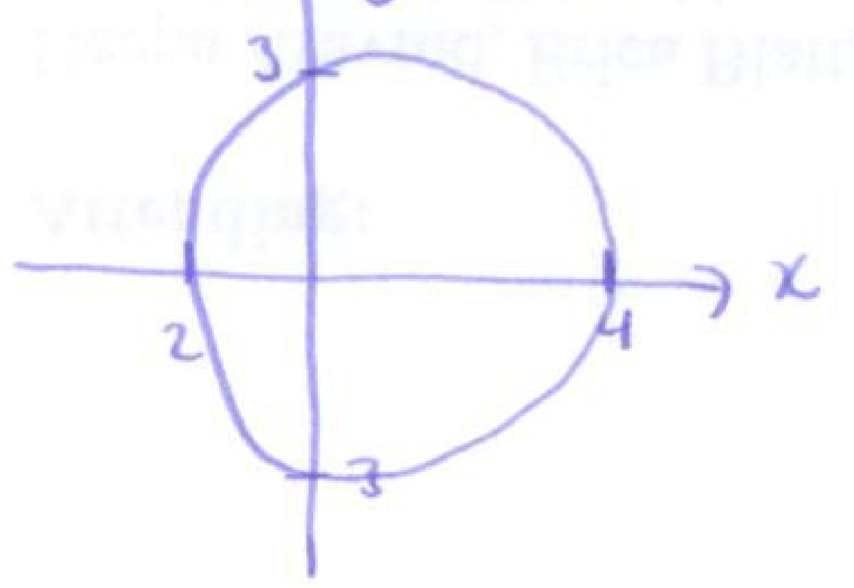
$f^{(n)}(x) = (-1)^n e^{-x}$ $\quad f^{(n)}(0) = (-1)^n \quad a_n = \dfrac{(-1)^n}{n!}$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots = \sum_{n=0}^\infty \frac{(-1)^n}{n!}x^n$$

b) ratio test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)!}x^{n+1}\frac{n!}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0$

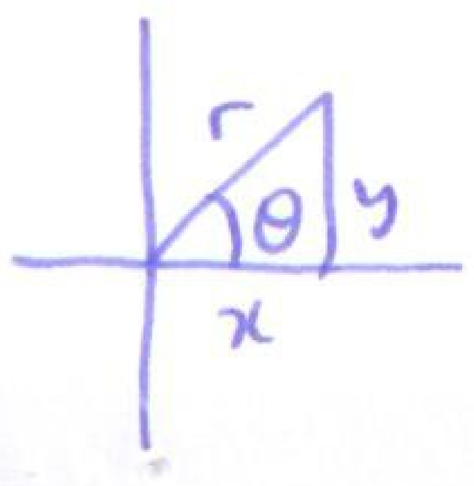
radius of convergence $R = \infty$.

Q3 a) $r = 3+\cos(\theta)$



b) $$\int_0^{2\pi} \frac{1}{2}\left(3+\cos\theta\right)^2 d\theta.$$

c)



$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$ $\quad x = (3+\cos(t))\,\overset{\cos}{\cancel{\sin}}\,t$

$\qquad\qquad\qquad y = (3+\cos(t))\sin t$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

$$x = (3+\cos(t))\cos(t)$$

$$\frac{dx}{dt} = (-\sin(t))\cos(t) + (3+\cos(t))(-\sin(t)).$$

$$y = (3+\cos(t))\sin t$$

$$\frac{dy}{dx} = (-\sin t)\sin t + (3+\cos(t)).\cos(t).$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{-1+0}{3.(-1)} = \frac{1}{3}.$$

Q4 a) $\int x + \frac{1}{x}\, dx = \frac{1}{2}x^2 + \ln|x| + c$

b)

$$\begin{array}{r|l} & x - 1 \\ x+1 & x^2+1 \\ & x^2+x \\ \hline & -x+1 \\ & -x-1 \\ \hline & 2 \end{array}$$

$$\int x - 1 + \frac{2}{x+1}\, dx = \frac{1}{2}x^2 - x + 2\ln|x+1| + c.$$

c) $\int \frac{x}{x^2+1}\, dx$ $\quad u = x^2+1$, $\frac{du}{dx} = 2x$

$$\int \frac{x}{u}\frac{dx}{du}\, du = \int \frac{x}{u}\frac{1}{2x}\, du = \frac{1}{2}\int \frac{1}{u} du$$

$$= \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+1| + c$$

d) $\int \frac{1}{x^2+1}\, dx = \tan^{-1}(x) + c$

Q5 $\left(\frac{2}{e}\right) < 1$ geometric series so $\sum_{n=0}^{\infty}\left(\frac{2}{e}\right)^n = \frac{1}{1-2/e}$ converges.

Q6 $\sum_{n=0}^{\infty}\frac{1}{1+n^2} < \sum_{n=0}^{\infty}\frac{1}{n^2}$ comparison test with p-series with $p=2$, converges.

Q7 Alternating series test $a_n = \frac{1}{1+n^2}$ $\quad a_n$ positive, decreasing, $a_n \to 0$ as $n \to \infty$ so converges.

Q8 ratio test: $\lim_{n\to\infty}\left|\frac{e^{n+1}}{(n+1)!}\frac{n!}{e^n}\right| = \lim_{n\to\infty}\left|\frac{e}{n+1}\right| = 0 < 1$ so converges.

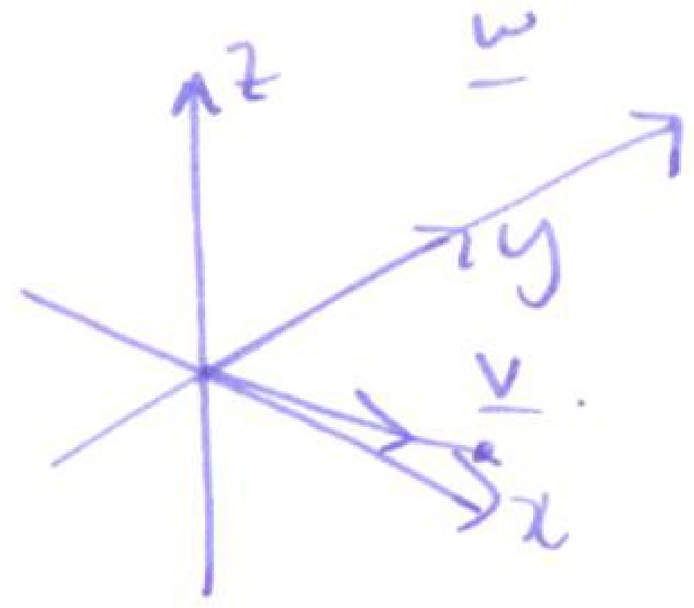
Q9 a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\log\left(1+\frac{1}{n}\right)}$

$\lim_{n\to\infty} n\log\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\log(1+1/n)}{1/n} \underset{\text{l'Hôpital}}{=} \lim_{n\to\infty}\frac{1/1+1/n \cdot -\frac{1}{n^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$

so $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1 = e$

b) No. $\sum_{n=0}^{\infty} a_n$ converges $\Rightarrow$ $a_n \to 0$ so $a_n \not\to 0 \Rightarrow \sum_{n=0}^{\infty} a_n$ does not converge.

Q10 a)



b) $\underline{r}(t) = \langle 2,3,-6\rangle + t\langle 3,1,-1\rangle$.

c) $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 1 & -1 \\ 2 & 5 & 1 \end{vmatrix} = \underline{i}\begin{vmatrix} 1 & -1 \\ 5 & 1\end{vmatrix} - \underline{j}\begin{vmatrix} 3 & -1 \\ 2 & 1\end{vmatrix} + \underline{k}\begin{vmatrix} 3 & 1 \\ 2 & 5\end{vmatrix}$

$= \langle 6, -5, 13\rangle$

d) $6x - 5y + 13z = 0$.