§8.8 Three reflections or two involutions

An involution is an isometry f such that $f^2 =$ identity.

i.e. : reflections, rotations by π.

Möbius group acts on upper halfspace

"boundary" $\mathbb{R} \cup \{\infty\}$.
S^1_∞.

Claim for any ~~four~~ three points $\{p, q, r\}$ in $\mathbb{R} \cup \{\infty\}$, there is a Möbius* map taking $\{p, q, r\}$ to $\{0, 1, \infty\}$.

Proof $f(z) = \lambda \dfrac{z-p}{z-r}$ $\qquad f(q) = \lambda \dfrac{q-p}{q-r} = 1$.

$$f(z) = \frac{(q-r)(\overset{(-)}{z}-p)}{(q-p)(\overset{(-)}{z}-r)}$$

(need $\bar{z}$ if $\det < 0$). works. $\square$ (show can take p, q, r to a, b, c).

Claim : this map f is unique! so the map that takes $\{a, b, c\}$ to $\{a', b', c'\}$ is unique!

Claim for any four points $\{p, q, r, s\}$ in $\mathbb{R} \cup \{\infty\}$ there is a Möbius* map taking them to $\{0, 1, \infty, c\}$ where c is the cross-ratio $\dfrac{(q-r)(s-p)}{(q-p)(s-r)}$.

claim for any four points $\{p, q, r, s\}$ in $\mathbb{R} \cup \{\infty\}$ there is a Möbius* map taking them to $\{q, p, s, r\}$.

Proof suffices to show we can send $\{0, 1, \infty, c\}$ to $\{1, 0, c, \infty\}$.

$$f(z) = c\frac{(z-1)}{(z-c)}$$ check $\square$.

<u>claim</u> If f is a Möbius$^+$ map that swaps two points, then f is an involution.

<u>Proof</u> suppose p, q are swapped, i.e. $f(p) = q$
$f(q) = p$

now let r be any point not fixed by ~~g~~ f so ~~g~~ $f(r) = s \neq r$.

now let g be the map which sends $\{p, q, r, s\}$ to $\{q, p, s, r\}$.

so $f = g$ by uniqueness of 3-point maps.

g is conjugate to $h(z) = \dfrac{c(z-1)}{z-c}$ $\quad h^2(z) =$ identity. (can compute directly or:).

g^2 takes $\{p, q, r, s\}$ to $\{p, q, r, s\}$ so is identity $\Rightarrow f = \mathrm{id}$. $\square$.

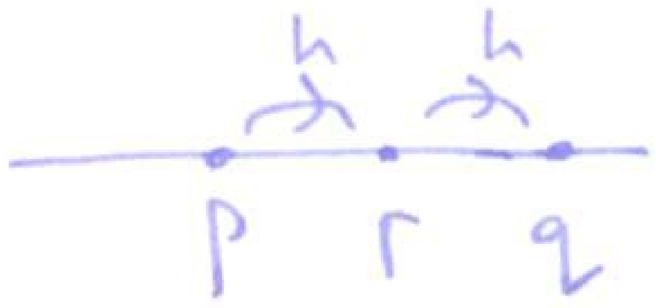
<u>claim</u> for any three points $\{p, q, r\}$ there is an involution which swaps p, q and fixes r.

<u>Proof</u>

$$\begin{tikzcd} \{p,q,r\} \arrow[r, "f\ \text{exists.}", "\text{unique}"'] \arrow[rr, bend right, "\text{identity.}"'] & \{q,p,r\} \arrow[r, "f", "\text{unique}"'] & \{p,q,r\} \end{tikzcd}$$

$\square$.

<u>Thm</u> (two involutions)
Any Möbius$^+$ map is a product of two involutions.



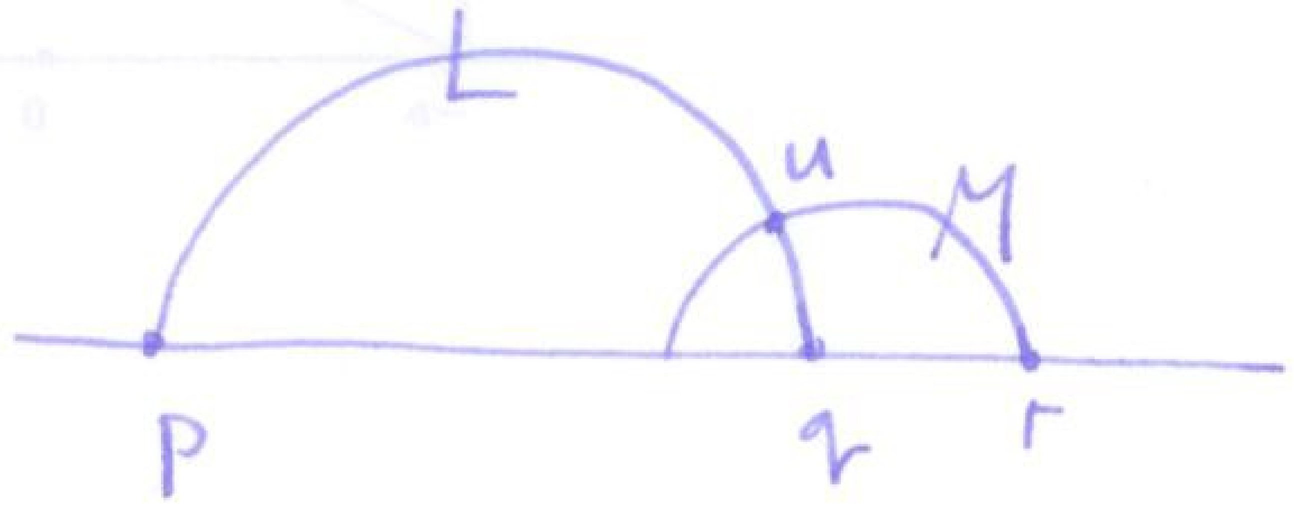
<u>Proof</u> Let $h \in \text{Möb}^+$. If $h =$ identity then involution.
$h \neq$ identity $\Rightarrow$ there is a point p s.t. $h(p) \neq p$, let $h(p) = r$
let $h(r) = q$. If $p = q$ then h involution by above, so assume $p \neq q$)

let $f : \{p, q, r\} \longmapsto \{q, p, r\}$ (involution) consider $fh : \left\{ \begin{array}{l} p \xmapsto{h} r \xmapsto{f} r \\ r \xmapsto{h} q \mapsto p \end{array} \right\}$ swaps p, r so fh is an involution, call it $g = fh$.

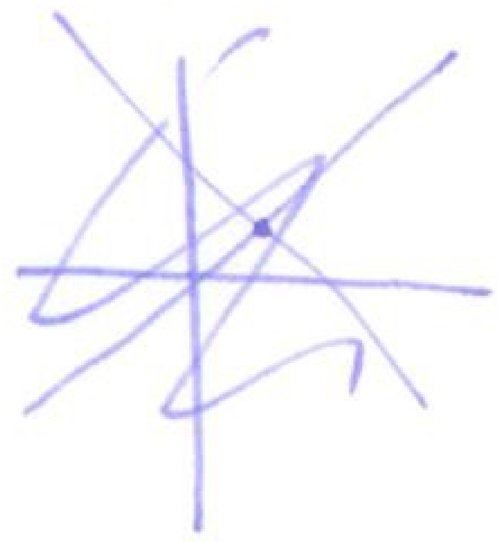
so $h = f^{-1} g$ (product of two involutions) $\square$.

<u>Thm</u> (three reflections)

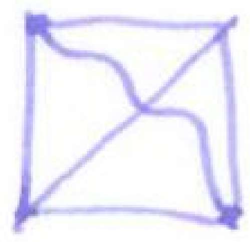
f as above: swaps p, q fixes r.



so preserves L, swaps endpoints,



so has a fixed point, u say

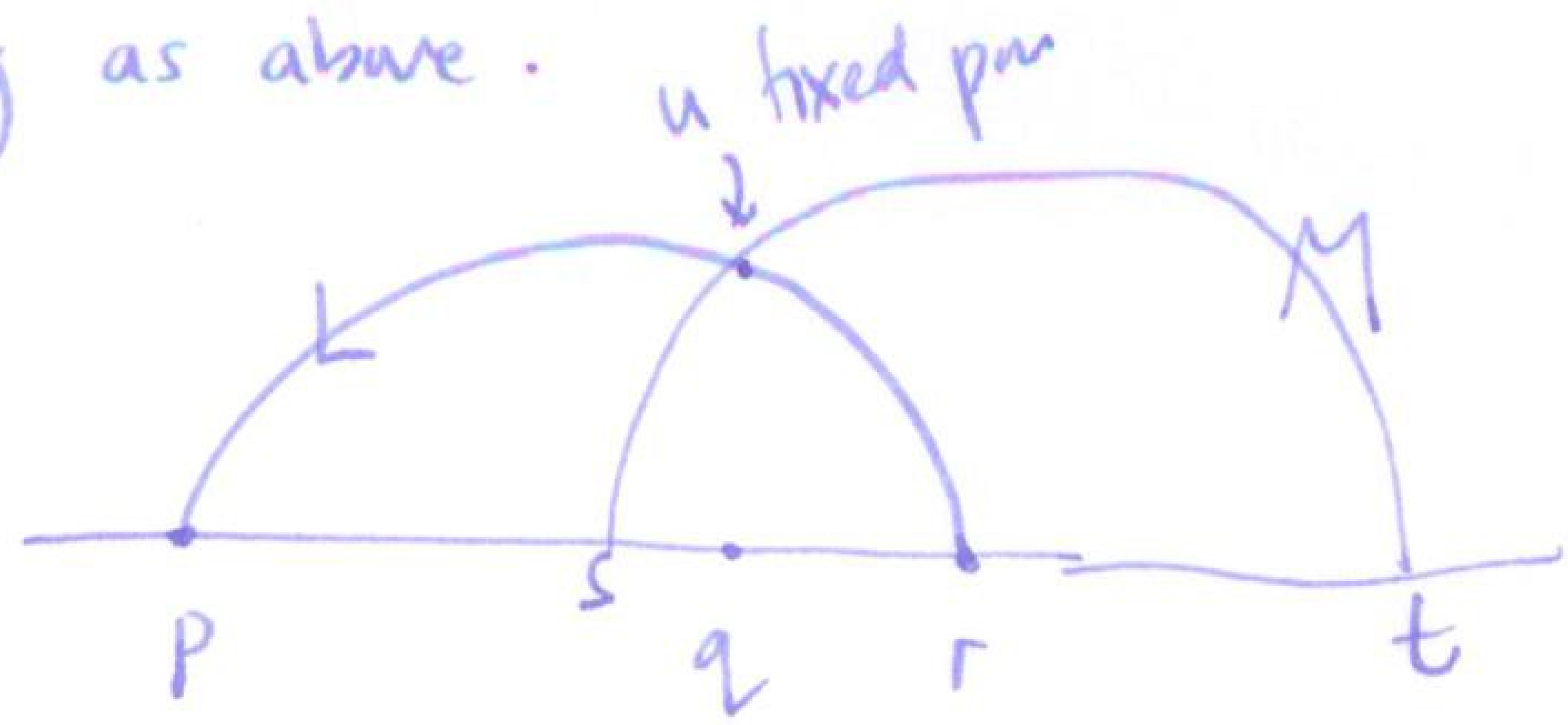


let M be the line from u to r, this is also fixed by f

Mob$^+$ preserves angles $\Rightarrow$ L, M meet at right angles.

so f has same effect on $\{p, q, r\}$ as reflection in M so f <u>is</u> reflection in M. (by uniqueness on three points).

g as above.



let M be perpendicular line through fixed point u.

if g has a fixed point in $\mathbb{R} \cup \{\infty\}$ then same argument as above, g is a reflection in M.

if g has no fixed points, then swaps s, t, so acts on $\{p, r, s\}$ as product of reflections in L and M $\square$.

§8.9 Other models for hyperbolic space

(read section §8.9)

(partial) models in $\mathbb{R}^3$.

curvature of surfaces in $\mathbb{R}^3$:


positive curvature (e.g S^2)


zero curvature (flat) $\mathbb{R}^2 \subset \mathbb{R}^3$


negative curvature (saddle shaped)

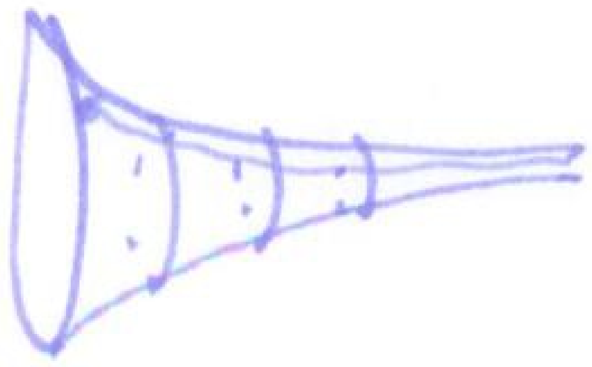
Q: is there a model for $\mathbb{H}^2$ in $\mathbb{R}^3$ (like $S^2 \subset \mathbb{R}^3$) **A**: No.

but there are partial models.

Example tractroid: surface of revolution of a tractrix.

tractrix:

curve such that the length of the tangent line to the x-axis is constant.



this surface has constant negative curvature, so is a local model, but it's not a disc, and it's not complete, i.e. can't extend geodesics indefinitely

The disc model

Isometries of $\mathbb{H}^2$: $\dfrac{az+b}{cz+d}$ $a,b,c,d \in \mathbb{R}$. ($SL_2\mathbb{R}$)

more generally $\dfrac{az+b}{cz+d}$ $a,b,c,d \in \mathbb{C}$ acts on $\mathbb{C}$. ($SL_2\mathbb{C}$)

preserves circles and lines. (and angles) in particular, there is a map from the upper half space to the disc

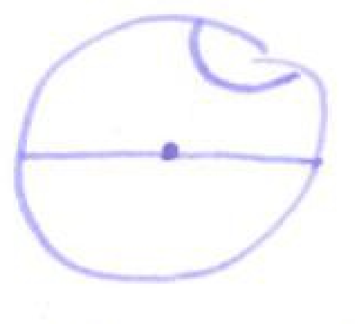
inverse.

$z \mapsto \dfrac{-zi+1}{z-i}$

$$z \mapsto \frac{-zi-1}{-z-i} = \frac{zi+1}{z+i}$$

hyperbolic lines

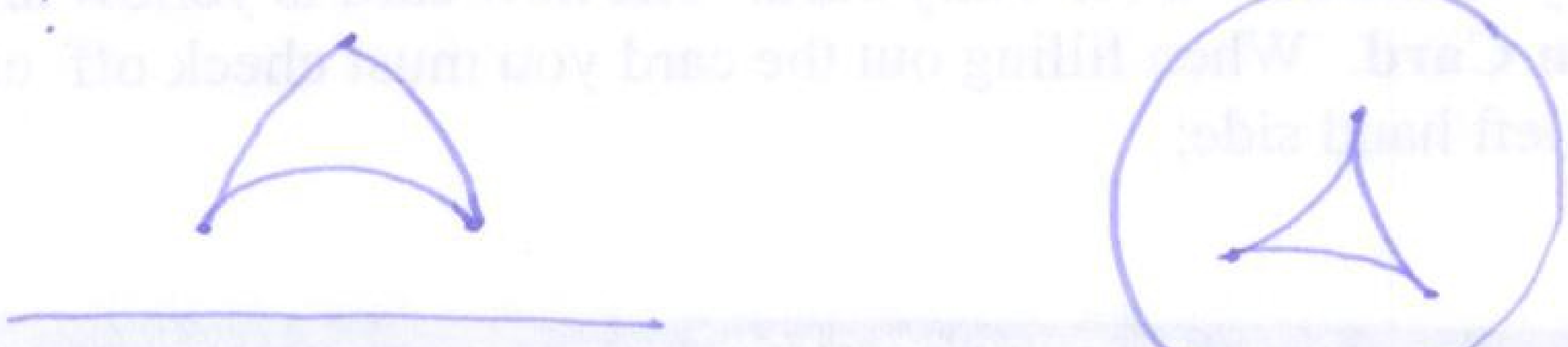
$\mapsto$



hyperbolic lines (see picture page 210)

↑ rotations around O are just Euclidean rotations.

Triangles in $\mathbb{H}^2$



sum of angles smaller than π!

Fact area of triangle in $\mathbb{H}^2$ is $\pi(\alpha+\beta+\gamma)$.

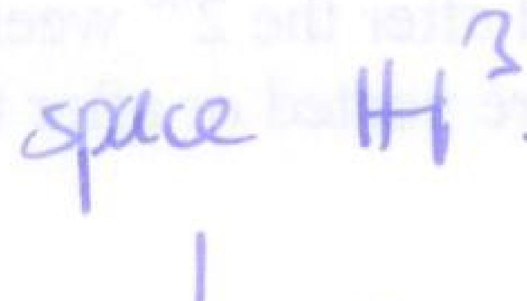
There is a "biggest" ideal triangle with area π. (angles zero)

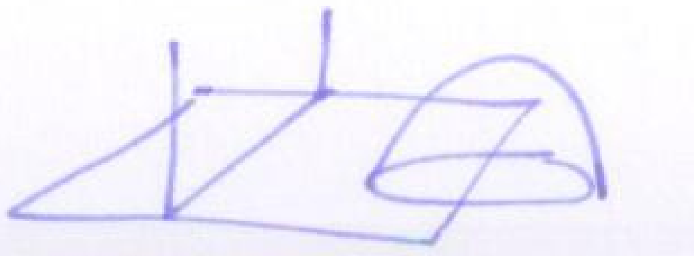
Q: how many ideal triangles are there up to isometry?

Q: what's the metric in the disc model?

Hyperbolic paper: exponential growth of area/circumference.

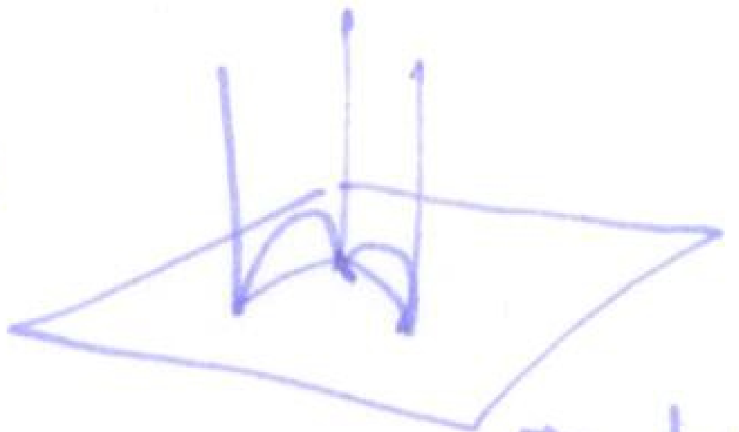
§15 [weeks] 3d hyperbolic space $\mathbb{H}^3$?

two models: upper half space  ball: 

planes: embedded copies of $\mathbb{H}^2$:  

$\mathrm{Isom}(\mathbb{H}^3) \cong PSL(2,\mathbb{C})$

polyhedra have "smaller" angles



$z \mapsto \dfrac{az+b}{cz+d}$ acts transitively on triples of points.

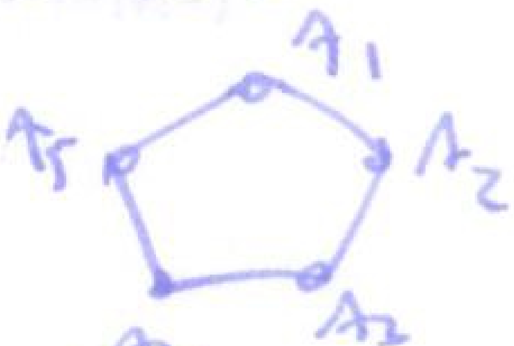
can take any three points to $\{0, 1, \infty\}$

can take any four points $\{p, q, r, s\}$ to $\{0, 1, \infty, c\}$ ← cross ratio.

volume of a hyperbolic tetrahedron is a function of the cross ratio.

§ (see website) <u>Euler's formula and regular polyhedra.</u> (in $\mathbb{R}^3$)

<u>Defn</u> A polygon is an (ordered) collection of points (vertices) $A_1, \dots, A_n$ and line segments A_1A_2, $A_2A_3, \dots, A_nA_1$ (edges) joining the points.



<u>Defn</u> A polygon whose vertices lie in a single plane is called <u>planar</u>.

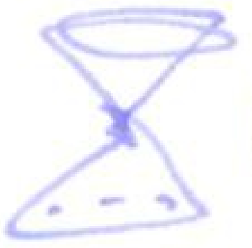
For this section: all polygons will be planar.

<u>Defn</u> A polygon is <u>equilateral</u> if all sides have the same length
<u>equiangular</u> if all angles are equal.

<u>Defn</u> A polygon is <u>regular</u> if it is equilateral and equiangular.

<u>Defn</u> A polyhedron is a solid whose surface consists of a number of polygonal faces:

- every side of every polygon belongs to exactly one other polygon.
- the faces that share a vertex form a chain of polygons

in which every consecutive pair ~~for~~ share a side.  bad

Def$^{\underline{n}}$ A polyhedron is <u>regular</u> if all faces are equal regular polygons, and the same number meet at every vertex.

$\{p,q\} \leftarrow$ means regular p-gons, q of which meet at each vertex.

Def$^{\underline{n}}$ A region is <u>convex</u> if every pair of points in the region is connected by a straight line segment contained in the region.

Def$^{\underline{n}}$ A region is <u>simple</u> if it has no holes, i.e. every closed curve can be continuously shrunk to a point in the region.

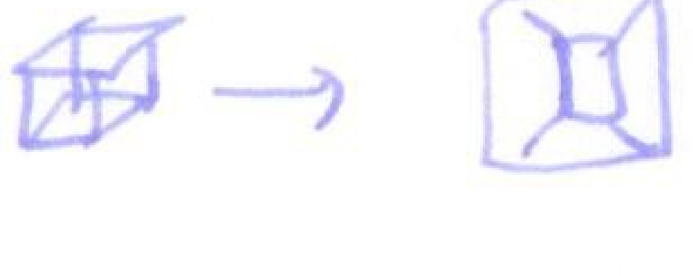
convexity $\Rightarrow$ simple

$\not\Leftarrow$

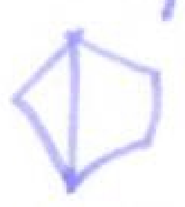
Th$^{\underline{m}}$ [Euler] For a simple polyhedron $F-E+V=2$.

$F=$ #faces, $E=$ #edges, $V=$ #verties.

Proof remove one face of the polydron, ~~and~~ the remainder is a planar graph (e.g. stereographic projection) (F, E, V unchanged)

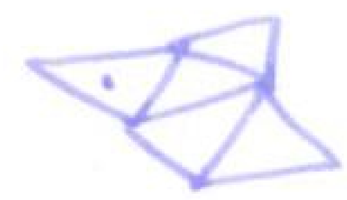


We can change the planar graph without changing $\chi = F-E+V$.

1. If there is a polygonal face with more than 3 sides, draw a diagonal  $\left(\begin{smallmatrix}+1\ E\\ +1\ F\end{smallmatrix}\right)$ doesn't change χ.

Repeat untill all ~~edg~~ faces are triangles.

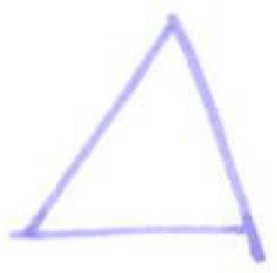
2. Remove one triangle with two edges shared by the exterior of the network $(-1\ V, -2E\ -1\ F)$



Repeat until no more.

3. Remove a triangle with only one edge adjacent to the exterior $(-1\ E, -1F)$

Repeat 2,3 until only one triangle left



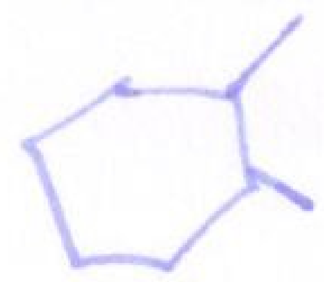
$V-E+F$
$3-3+2=2$.

Therefore $\chi = 2$ $\square$.

Thm There are only 5 regular convex polyhedra.

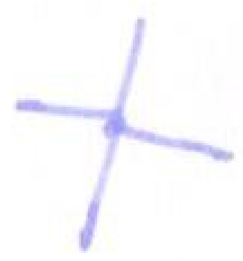
Proof Suppose there is a $\{p,q\}$ polyhedron.

regular p-gons.



every edge shared by two faces.

$$pF = 2E$$



every vertex meets q edges, each edge has two ends.

$$qV = 2E$$

χ: $V-E+F=2$

$$\frac{2E}{q} - E + \frac{2E}{p} = 2$$

$$\frac{1}{p}+\frac{1}{q} = \frac{1}{2}+\frac{1}{E}$$

$p \geq 3$, $q \geq 3$. if both $p,q \geq 4$ then LHS $\leq \frac{1}{2}$ #

so at least one of $p,q = 3$.

if $p=3$: $\frac{1}{q} = \frac{1}{6} + \frac{1}{E}$ so $q = 3,4,5$

then $E = 6,12,30$

if $q=3$: $\frac{1}{p} = \frac{1}{6} + \frac{1}{E}$ so $p = 3,4,5$

then $E = 6,12,30$

Possibilities

	F	E	V	
$\{3,3\}$	4	6	4	tetrahedron 
$\{3,4\}$	8	12	6	octahedron 
$\{4,3\}$	6	12	8	cube 
$\{3,5\}$	20	30	12	Icosahedron
$\{5,3\}$	12	30	20	Dodecahedron