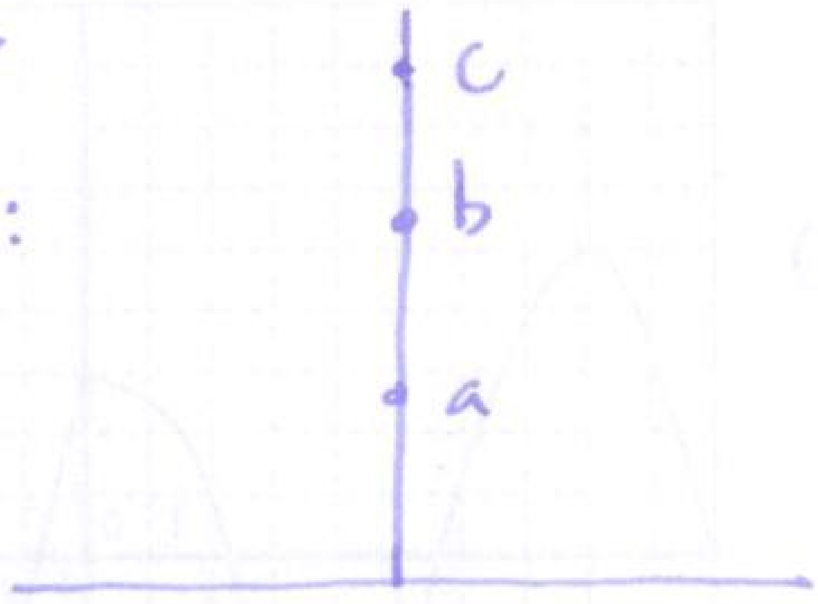
so as $\epsilon \to 0$ $f(z+\Delta z) \to f(z)$ along direction $\theta + \text{const}(z)$.

so

difference of angles is preserved, as required. $\square$.

§8.6 Hyperbolic distance

distance on the y-axis:



$$d(a,b) = \left|\log \frac{a}{b}\right|$$

(note: $\log \frac{a}{b} = -\log \frac{b}{a}$.

note: $d(a,a) = \left|\log \frac{a}{a}\right| = |\log 1| = 0$

$d(a,b) = 0 \Rightarrow \left|\log \frac{a}{b}\right| = 0 \Rightarrow \log \frac{a}{b} = 1 \Rightarrow a = b$.

$d(a,b) + d(b,c) = \left|\log \frac{b}{a}\right| + \left|\log \frac{c}{b}\right| = \log \frac{b}{a} + \log \frac{c}{b} = \log \frac{bc}{ab} = \left|\log \frac{c}{a}\right|$.

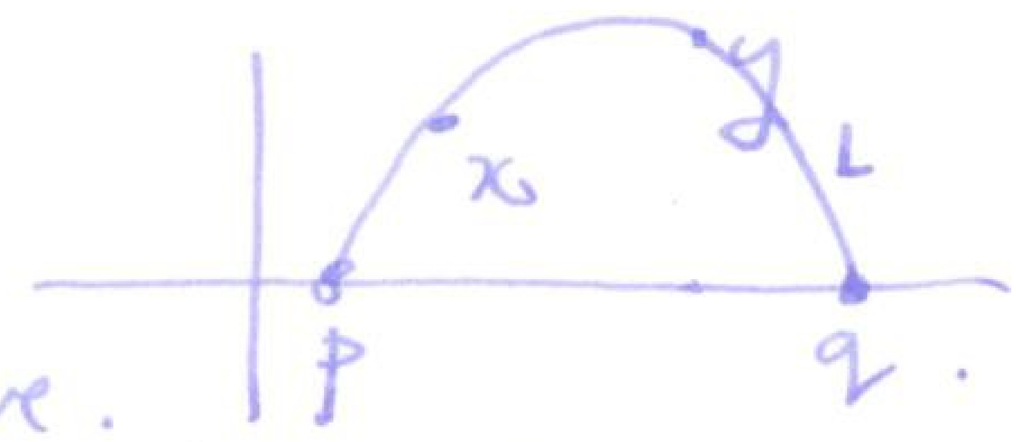
$= d(a,c)$

invariant under $z \mapsto az$: $d(x,y) = \left|\log \frac{x}{y}\right|$.

$d(ax,ay) = \left|\log \frac{ax}{ay}\right| = \left|\log \frac{x}{y}\right| = d(x,y)$.

$z \mapsto -\frac{1}{z}$: $d\left(-\frac{1}{x}, -\frac{1}{y}\right) = \left|\log \frac{-1/x}{-1/y}\right| = \left|\log \frac{x}{y}\right| = d(x,y)$.

distance between arbitrary points:



map L to y-axis, measure distance there.

claim: there is a Möbius map taking $p \mapsto 0$ and $q \mapsto \infty$.

proof: $z \mapsto \frac{z-p}{z-q}$ works. $\square$.

$$\cosh d(x,y) = 1 + \frac{|x-y|^2}{2 x_n y_n}$$

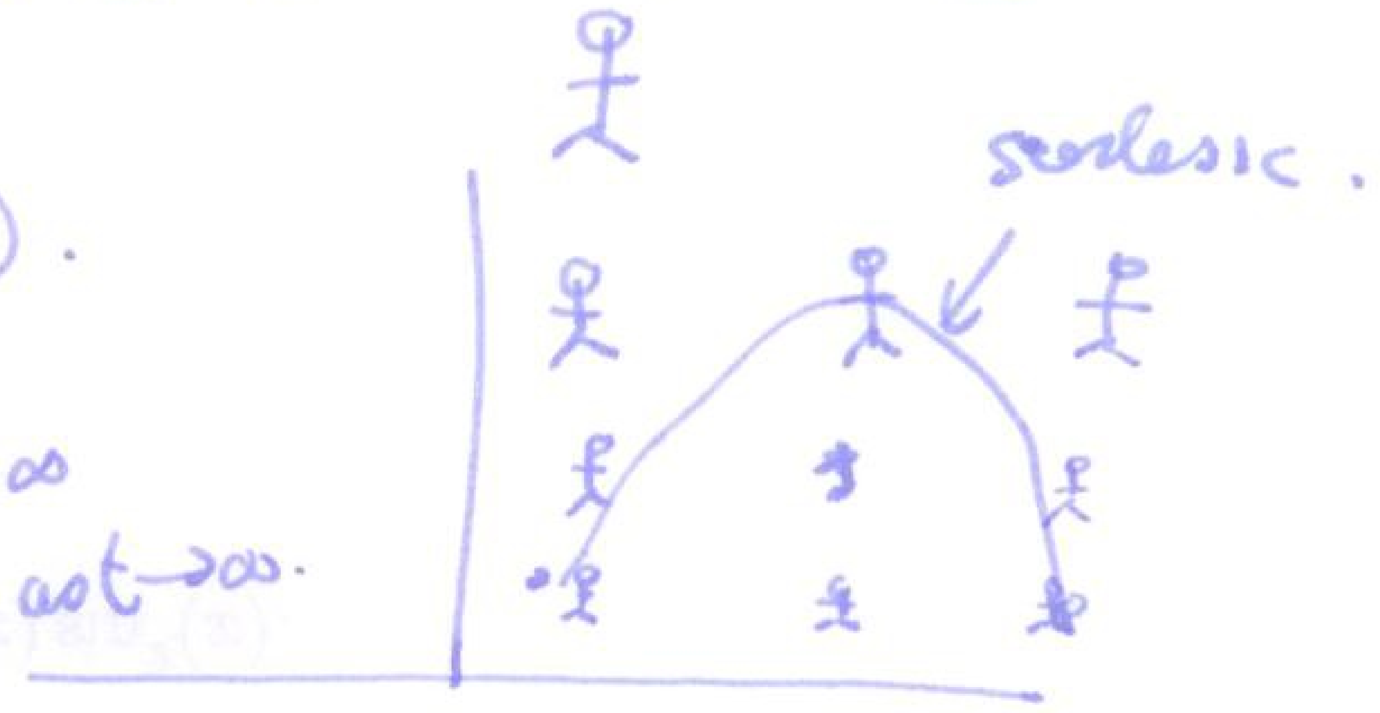
<u>claim</u> this def^n of distance does not depend on choice of map $L \to$ y-axis.

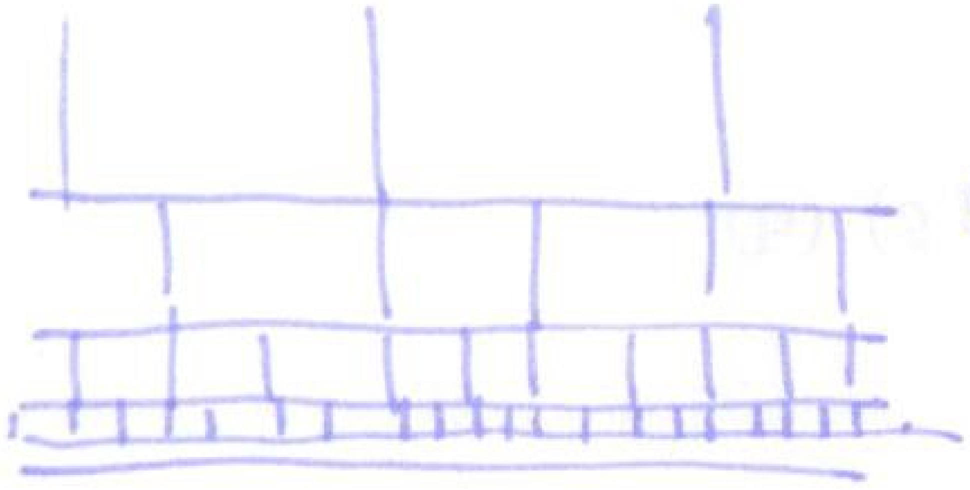
<u>proof</u>: any two choices differ by a Möbius map that preserves the y-axis. spse $f(z) = \dfrac{az+b}{cz+d}$ and $f(0)=0$, $f(\infty)=\infty$. then $\frac{b}{d} = 0 \Rightarrow b=0$ & $\frac{a}{c} = \infty$ i.e. $c=0$.

(or: want $cz+d=0$ to have <u>no</u> sol^{ns})

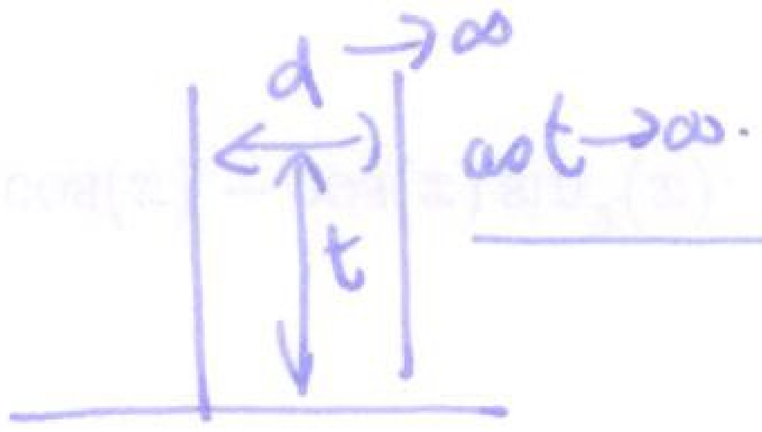
so $f(z) = \dfrac{az}{d} = az$. but distance preserved under $z \mapsto az$ $\square$.

distance in the upper half space (picture p.195).



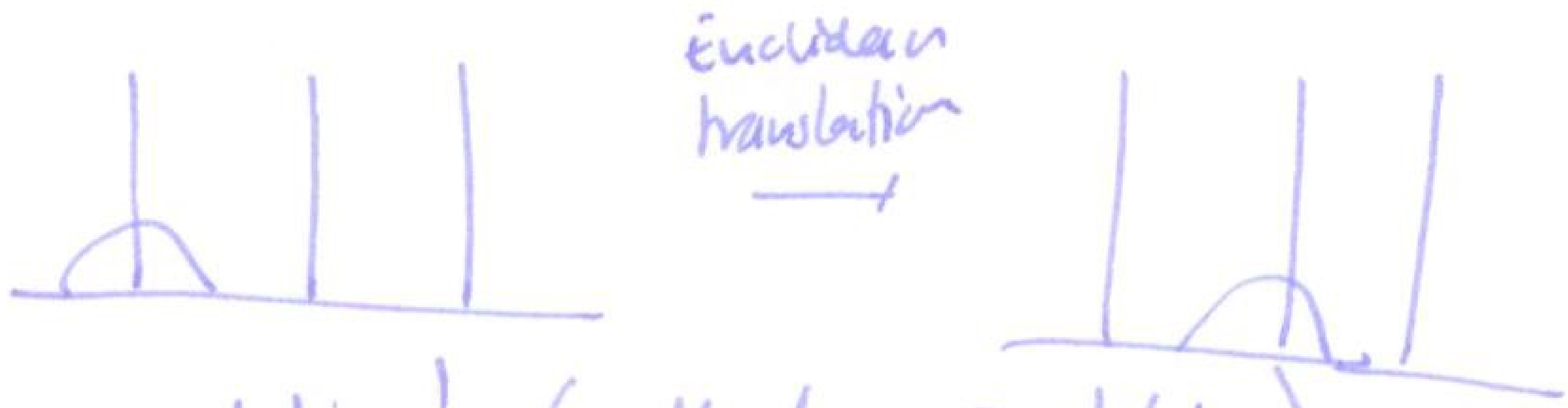


$\in$ all pentagons same size ($z \mapsto 2z$).



§8.7 Hyperbolic translations and rotations

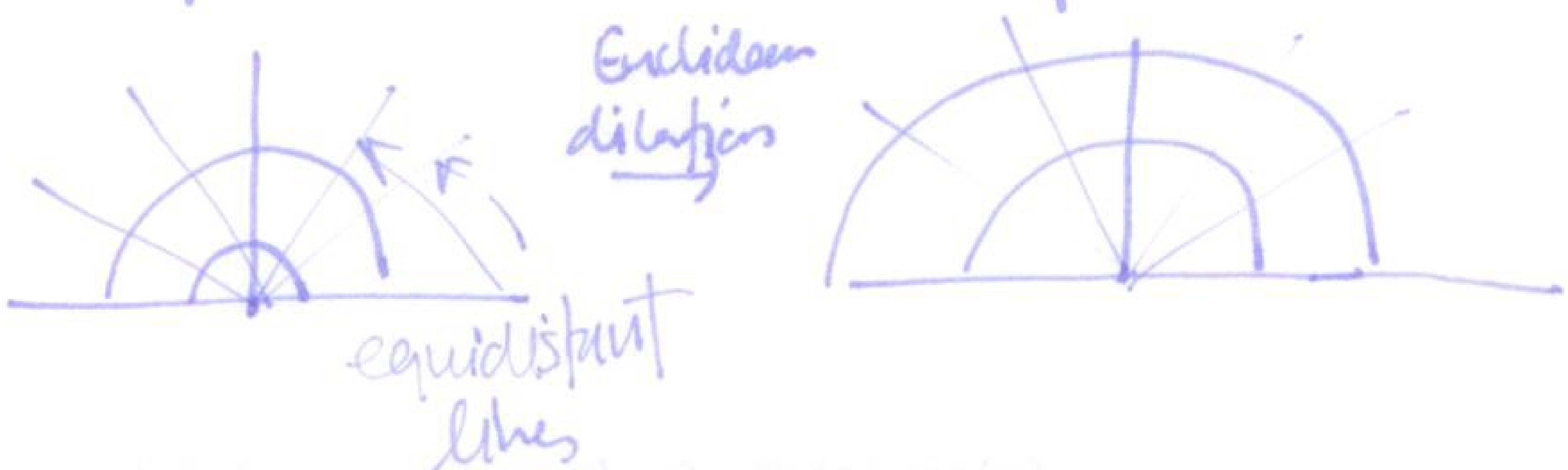
$z \mapsto z+a$.



<u>not</u> a translation or rotation! (called a parabolic)

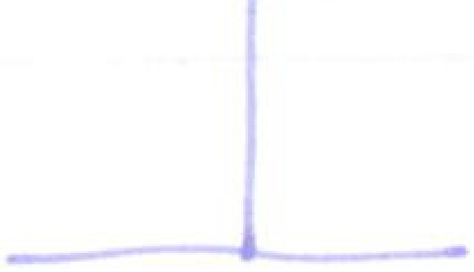
unique fixed point ∞. moves points arbitrarily small distance!

$z \mapsto az$



corresponds to hyperbolic translation along y-axis.

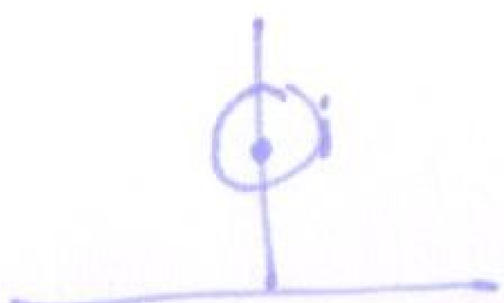
$z \mapsto -\bar{z}$



reflection in y-axis.

fixed point: $z = -\frac{1}{z}$. $z^2 = -1$ $z = \pm i$.

$z \mapsto -\frac{1}{z}$



rotation by π about i. (fixes circles about i).